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AK Type Production Function in DSGE Model†

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Abstract

Some DSGE (Dynamic Stochastic General Equilibrium) models include no consideration of long-run economic growth. Our paper presents consideration of a DSGE model with economic growth in the long run. As shown by the data, economic growth continues in terms of a long span. Therefore, we consider that it is appropriate to examine short-run and long-run policy effects on macroeconomic variables in a model in which long-run economic growth continues.

The contribution represented by our paper is the description of the simple endogenous growth DSGE model. Although there exist some related papers about endogenous growth DSGE models, our setting is a very simple DSGE model, showing the ease of setting a DSGE model with endogenous growth.

Key words: DSGE model, Endogenous growth
JEL Classification Code: E60

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1. Introduction

Many DSGE (Dynamic Stochastic General Equilibrium)-model related papers include an assumption of a neoclassical product function. Our paper sets a DSGE model with an AK-type production function in the intermediate goods sector. Considering an AK-type product function as the endogenous growth model, we can derive the equilibrium of the model economy as the balanced growth path.

Based on the standard type of DSGE described by Kato (2008) and Eguchi (2011), we set a DSGE model with an AK-type production function as endogenous growth model. Some model settings exist for endogenous growth models. As reported by Barro (1990), Futagami, Morita and Shibata (1993), public investment can be considered. Lucas (1988) considers human capital accumulation and derives the equilibrium in an endogenous growth model. Moreover, Romer (1986) considers externality of the capital stock and consequently derives the endogenous growth model.

As shown by the reports of the related literature presented above, many studies have examined endogenous growth in the field of macroeconomics. Nevertheless, few papers describe a DSGE model with an endogenous growth model. The aim of this study is to set a DSGE model that is as simple as possible, but with endogenous growth. To set a simple DSGE model with endogenous growth, we consider a Grossman and Yanagawa (1993) type production function. They assume that labor productivity depends on the physical capital stock per capita. Then we can readily derive an AK-type production function as endogenous growth model. Moran and Queralto (2018) and Queralto (2020) derive an endogenous growth model with a micro foundation of innovation.

Some DSGE models do not consider long-run economic growth. Our paper presents consideration of a DSGE model with long-run economic growth. As shown by the data, the economics growth continues in terms of the long span. Therefore, we consider that it is appropriate to examine short-run and long-run effects of policies on the variables of macroeconomics in model that economic growth continues in the long run.

The remainder of this paper is shown as follows. Section 2 sets the model and derives the equilibrium of the model economy. Section 3 shows the balanced growth path equilibrium. Section 4 shows how to derive the parameters with calibration; section 5 presents the example of simulation. The final section concludes our paper.

2. Model

This model economy includes agents of three types: households, firms, and government. Herein, we explain the behavior of each agent. The setting is based on work by Hayashida, Yasuoka, Nanba and Ono (2018). The different part of the setting is about the adjustment cost of investment and the AK-type production function.

2.1. Household

The population size is assumed to be unity, with no population growth. For this study, the following household utility function u_t in t period is assumed:

$$u_t = \frac{c_t^{1-\theta}}{1-\theta} + \frac{m_t^{1-\mu}}{1-\mu} - \frac{l_t^{1+\kappa}}{1+\kappa}, 0 < \theta, 0 < \mu, 0 < \kappa. \quad (1)$$

In that equation, c_t denotes consumption, m_t represents the real money stock, and l_t represents the labor supply time.

The household has a unit of time: $1 - l_t$ represents leisure time. Also, θ, μ and κ are parameters that are given exogenously. The budget constraint for t period is given as the following.

$$m_t + b_t + c_t + I_t = \frac{1}{1+\pi_t}[(1+i_t) + m_{t-1}] + \varphi_t + w_t l_t + r_t K_{t-1} \tag{2}$$

In that equation, b_t stands for bonds as the riskless asset, K_t denotes physical capital stock, I_t represents investment for physical capital stock, φ_t denotes the profit of firms, π_t expresses inflation rate, i_t signifies the nominal interest rate, r_t is the real interest rate, and w_t is the wage rate.

The household owns the firm and obtains the profit. Defining p_t as the price index, the inflation rate is given as $1 + \pi_t = \frac{p_t}{p_{t-1}}$. The household owns the physical capital stock, which is lent to firms; the rental rate is given as r_t.

The dynamics of physical capital stock K_t is given as

$$K_t = I_t + (1-\delta)K_{t-1} - S\left(\frac{\frac{I_t}{K_{t-1}}}{\frac{I_{t-1}}{K_{t-2}}}\right) I_t, 0 < \delta < 1. \tag{3}$$

Therein, δ is the depreciation rate. Also, $S\left(\frac{I_t/K_{t-1}}{I_{t-1}/K_{t-2}}\right)$ is the adjustment cost of investment. We assume that $S' > 0, S(1) = S'(1) = 0$. If the investment is larger, then the adjustment cost is higher.

Now, we derive the household optimal allocations to maximize the following lifetime utility U subject to the budget constraint and the dynamics of physical capital.

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\theta}}{1-\theta} + \frac{m_t^{1-\mu}}{1-\mu} - \frac{l_t^{1+\kappa}}{1+\kappa}\right], 0 < \beta < 1. \tag{4}$$

In that equation, E_0 stands for the expectation operator; β represents the discount factor.

Because of first-order condition of the household maximizing problem, the following Euler equation of consumption can be presented as

$$c_t^{-\theta} = \beta E_t c_{t+1}^{-\theta} \frac{1 + i_{t+1}}{1 + \pi_{t+1}}. \tag{5}$$

We can derive the following equation by multiplying K_{t-1}^{θ} at both sides of (5) as

$$\left(\frac{c_t}{K_{t-1}}\right)^{-\theta} = \beta E_t (1+g_t)^{-\theta} \left(\frac{c_{t+1}}{K_t}\right)^{-\theta} \frac{1 + i_{t+1}}{1 + \pi_{t+1}}, \tag{6}$$

where $1 + g_t = \frac{K_t}{K_{t-1}}$. Additionally, we can obtain

$$E_t\big(r_{t+1} + q_{t+1}(1-\delta)\big) = q_t E_t \frac{1 + i_{t+1}}{1 + \pi_{t+1}}, \tag{7}$$

where q_t denotes Tobin's q and is given as $q_t = \frac{\gamma_t}{\lambda_t}$. Also, λ_t and γ_t respectively represent the Lagrange multipliers of the constraints (2) and (3). The investment to maximize household utility can be shown as

$$1 = q_t \left(1 - S\left(\frac{\frac{I_t}{K_{t-1}}}{\frac{I_{t-1}}{K_{t-2}}} \right) - S'\left(\frac{I_t/K_{t-1}}{I_{t-1}/K_{t-2}} \right) \left(\frac{I_t/K_{t-1}}{I_{t-1}/K_{t-2}} \right) \right) + E_t q_{t+1} \frac{1+\pi_{t+1}}{1+i_{t+1}} S'\left(\frac{I_{t+1}/K_t}{I_t/K_{t-1}} \right) \left(\frac{I_{t+1}/K_t}{I_t/K_{t-1}} \right)^2 \frac{K_{t-1}}{K_t}. \tag{8}$$

The labor supply to maximize household utility can be shown as the marginal substitution rate of leisure and consumption as

$$w_t = \frac{l_t^{\kappa}}{c_t^{-\theta}}. \tag{9}$$

2.2. Firm

This model includes firms of two types: one for a final goods production firm and the other for an intermediate goods production firm. At the final goods sector, the production function is given by the constant elasticity of substitution form (CES) in which the intermediate goods are inputted. At the intermediate goods sector, the goods are produced by the input of labor and capital stock. The production function is of the AK type form.

2.2.1. Final Goods Firm

The final goods are assumed to be produced for the perfectly competitive market. We assume the following production function as

$$Y_t = \left(\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}, 1 < \varepsilon. \tag{10}$$

Therein, Y_t represents final goods, and Y_{jt} denotes intermediate goods produced by the j-th firm. ε denotes the elasticity of substitution.

The profit of final goods firm π_t^f is given as shown below:

$$\pi_t^f = p_t Y_t - \int_0^1 p_{jt} Y_{jt} dj, 0 \le j \le 1. \tag{11}$$

In that equation, p_{jt} represents the j-th intermediate goods price

Profit maximization derives the following demand function for intermediate goods as

$$Y_{jt} = \left(\frac{p_{jt}}{p_t} \right)^{-\varepsilon} Y_t, \tag{12}$$

and price index p_t and revenue $p_t Y_t$ as

$$p_t = \left(\int_0^1 p_{jt}^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}, \tag{13}$$

$$p_t Y_t = \int_0^1 p_{jt} Y_{jt}\, dj. \tag{14}$$

2.2.2. Intermediate Goods Firm

The intermediate goods of j-th firm are produced by inputting physical capital stock and labor. The intermediate goods production function is assumed as

$$Y_{jt} = K_{jt-1}^{\alpha}\left(A_{jt}N_{jt}\right)^{1-\alpha}, 0 < \alpha < 1. \tag{15}$$

In that equation, K_{jt} denotes the physical capital stock of j-th firm; N_{jt} stands for the labor input of j-th firm.

We assume $A_{jt} = a\frac{K_{jt-1}}{N_{jt}}$ $(0 < a)$, which is assumed by Grossman and Yanagawa (1993) to consider the AK-type production function, which is considered as the labor productivity. Substituting $A_{jt} = a\frac{K_{jt-1}}{N_{jt}}$ into (15), we obtain $Y_{jt} = a^{1-\alpha}K_{jt-1}$.

Defining the total production cost of j-th firm as $C_j = w_{jt}N_{jt} + r_{jt}K_{jt}$, we can consider the following Lagrange equation to minimize the total cost subject to production function (15):

$$\Lambda = w_{jt}N_{jt} + r_{jt}K_{jt-1} + \omega_{jt}\left(Y_{jt} - K_{jt-1}^{\alpha}\left(A_{jt}N_{jt}\right)^{1-\alpha}\right). \tag{16}$$

In that equation, ω_{jt} is the Lagrange multiplier of (15), w_{jt} denotes the wage rate of j-th firm, and r_{jt} represents the rental rate of the j-th firm.

Then, the following the relation between factor price and marginal productivity can be derived as

$$w_{jt} = \omega_{jt}(1-\alpha)a^{1-\alpha}\frac{K_{jt-1}}{N_{jt}}, \tag{17}$$

$$r_{jt} = \omega_{jt}\alpha a^{1-\alpha}. \tag{18}$$

Considering a constant returns to scale production function, (17) and (18), the total cost can be shown as

$$C_j = w_{jt}N_{jt} + r_{jt}K_{jt-1} = \omega_{jt}Y_{jt}. \tag{19}$$

With (12) and (19), the profit function of j-th firm can be shown as presented below:

$$\pi_{jt} = \frac{p_{jt}}{p_t}\left(\frac{p_{jt}}{p_t}\right)^{-\varepsilon}Y_t - \omega_{jt}\left(\frac{p_{jt}}{p_t}\right)^{-\varepsilon}Y_t. \tag{20}$$

For profit maximization, p_{jt} is set such that the following equation holds:

$$\omega_{jt} = \frac{\varepsilon - 1}{\varepsilon}\frac{p_{jt}}{p_t}. \tag{21}$$

2.2.3. Sticky Price

As does the model of Calvo (1983), we assume the sticky pricing model. Calvo (1983) considers the monopolistic competitive market and assumes that some firms can not set the optimal price at a probability.[1] The log form of optimal price is given by (21) as

$$\ln p_t^* = \ln\frac{\varepsilon}{\varepsilon - 1} + \ln\omega_t + \ln p_t. \tag{22}$$

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[1] Price-setting of Calvo (1983) is popular in the DSGE model. However, there exists the other type of price setting shown by Rotemberg (1982). Price setting of Rotemberg (1982) assumes the adjustment cost to change the price.

If we assume that the share ρ of firms can set the optimal price level and that the share $1-\rho$ of firms can not set the optimal price level, then we can obtain the following equation:

$$\ln(1+\pi_t)=E_t\ln(1+\pi_{t+1})+\frac{\rho^2}{1-\rho}\left(\ln\frac{\varepsilon}{\varepsilon-1}+\ln\omega_t\right). \tag{23}$$

2.3 Monetary Policy

The monetary policy in this paper is based on the following Taylor rule, as

$$\tilde{\imath}_t=\chi\tilde{\imath}_{t-1}+(1-\chi)\{\phi_1 E_t\tilde{\pi}_{t+1}+\phi_2\tilde{g}_t\}, 0<\chi<1, 0<\phi_1, 0<\phi_2. \tag{24}$$

In that equation, $\tilde{g}_t$ stands for the change level of g_t, $\tilde{\imath}_t$ denotes the change level of i_t, and $\tilde{\pi}_{t+1}$ signifies the change level of π_{t+1}.

This monetary policy depends on the change level of expectation of inflation $E_t\tilde{\pi}_{t+1}$, the change of growth $\tilde{g}_t$, and the previous nominal interest rate $\tilde{\imath}_{t-1}$.

3. Equilibrium

In this section, we derive the equilibrium of our model economy.

• Euler equation of consumption

Log linearization of (6) shows the following equation,[2] as

$$\hat{c}_t=\tilde{g}_t+\hat{c}_{t+1}-\frac{1}{\theta}E_t\tilde{\imath}_{t+1}+\frac{1}{\theta}E_t\tilde{\pi}_{t+1}. \tag{25}$$

Therein, $\hat{c}_t$ represents the rate of change of $\frac{c_t}{K_{t-1}}$.

• Fisher Equation

Log linearization of (7) shows the following equation.

$$E_t\hat{q}_{t+1}=\frac{1}{1-\delta}\left(\frac{1+i}{1+\pi}\left(\hat{q}_t+E_t(\tilde{\imath}_{t+1}-\tilde{\pi}_{t+1})\right)-\frac{r}{q}\hat{r}_t\right) \tag{26}$$

In that equation, $\hat{q}_t$ denotes the rate of change of q_t, $\hat{r}_t$ stands for the rate of change of r_t, π signifies the inflation rate in the steady state, i expresses the nominal interest rate in the steady state, r is the real interest rate in the steady state, and q represents q_t in the steady state.

• Labor Supply

Considering the household maximization problem, one can obtain the labor supply (9) as the marginal rate of substitution between leisure and consumption. By multiplying K_{t-1} at both sides of this equation and log linearization and assuming $\theta=1$, we can obtain the following equation, [3]

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[2] Our paper considers the variables divided by K_{t-1}. However, because of $Y_t=a^{1-\alpha}K_{t-1}$, these variables are the same with the variable divided by Y_t.

[3] Because of an increase in the disutility of labor, κ should be $\kappa>1$. With the log utility function of consumption, we obtain $\theta=1$. We note that $\tilde{g}$ is the gap of the level of balanced growth path. Then, at the balanced growth path, the gap of income growth rate does not affect labor–leisure choice. At the balanced growth path, we obtain $\kappa\hat{l}_t+\theta\hat{c}_t=0$. However, considering the balanced growth path, $\hat{l}_t=0$ and $\hat{c}_t=0$ are given at the balanced growth path. As the setting the GHH preference is set by Greenwood, Hercowitz and Huffman (1988). The utility function is assumed as $u_t=u(c_t-l_t)$. This setting holds the appropriate equilibrium to consider $c_t-l_t>0$. However, this setting does not always prevent violation of $l_t<1$.

$$\widehat{w}_t = \kappa \hat{l}_t + \hat{c}_t \tag{27}$$

In this equation, $\widehat{w}_t$ denotes the rate of change of $\frac{w_t}{K_{t-1}}$, and $\hat{l}_t$ stands for the rate of change of l_t.

• Investment

Log linearization of (8) yields the following equation.

$$\hat{I}_t = \frac{1+i}{2+i+\pi}\hat{I}_{t-1} + \frac{1+i}{2+i+\pi}E_t\hat{I}_{t+1} + \frac{1+i}{(2+i+\pi)S''(1)}\hat{q}_t \tag{28}$$

In that equation, $\hat{I}_t$ represents the rate of change of $\frac{I_t}{K_{t-1}}$.

• Dynamics of physical capital stock

Log linearization of (3) produces the following equation.

$$\tilde{g}_t = \frac{I}{K}\hat{I}_t \tag{29}$$

• Factor price

Log linearization of (17) and (18) shows the following equations, respectively, as presented below.

$$\widehat{w}_t = \widehat{\omega}_t - \hat{l}_t \tag{30}$$

$$\hat{r}_t = \widehat{\omega}_t \tag{31}$$

Therein, $\widehat{w}_t$ denotes the rate of change of $\frac{w_t}{K_{t-1}}$. Also, $\widehat{\omega}_t$ represents the rate of change of ω_t.

• New Keynesian Phillips Curve

Log linearization of (23) shows the following equation:

$$\tilde{\pi}_t = E_t\tilde{\pi}_{t+1} + \frac{\rho^2}{1-\rho}\widehat{\omega}_t. \tag{32}$$

• Monetary policy

$$\tilde{\imath}_t = \chi\tilde{\imath}_{t-1} + (1-\chi)\{\phi_1 E_t\tilde{\pi}_{t+1} + \phi_2\tilde{g}_t\} \tag{24}$$

• Goods market

The equilibrium condition of the goods market is $Y_t = c_t + I_t$. By dividing by K_{t-1} and using log linearization, we can obtain the following equation:

$$0 = \frac{c/K}{Y/K}\hat{c}_t + \frac{I/K}{Y/K}\hat{I}_t, or\ 0 = \frac{c}{K}\hat{c}_t + \frac{I}{K}\hat{I}_t. \tag{33}$$

In those equations, $\hat{Y}_t$ stands for rate of change of $\frac{Y_t}{K_{t-1}}$, Y/K expresses $\frac{Y_t}{K_{t-1}}$ in the steady state, I/K denotes $\frac{I_t}{K_{t-1}}$ in the steady state, and c/K signifies $\frac{c_t}{K_{t-1}}$ in the steady state.

Because of production function $Y_t = K_{t-1}^{\alpha}(A_tN_t)^{1-\alpha}$ and productivity $A_t = a\frac{K_{t-1}}{N_t}$, we obtain $Y_t = a^{1-\alpha}K_{t-1}$. Then, $\hat{Y}_t$ is always zero because of AK type model. The rate of change and the change level show the difference from the steady state value.

• Productivity shock

Our paper presents consideration of productivity shock as income uncertainty. We assume $A_t = (a_t + \bar{a})\frac{K_{t-1}}{N_t}$.

The shock is given as

$$\hat{a}_t = \phi_3 \hat{a}_{t-1} + f, 0 < \phi_3 < 1. \tag{34}$$

Therein, $\hat{a}_t$ denotes the rate of change of a_t; f stands for an exogenous shock.
Then, (17) and (18) change to the following expressions.

$$\widehat{w}_t = \widehat{\omega}_t + (1-\alpha)\hat{a}_t - \hat{l}_t, \tag{35}$$

$$\hat{r}_t = \widehat{\omega}_t + (1-\alpha)\hat{a}_t. \tag{36}$$

4. Calibration

We can estimate the model parameters with calibration. As an example, we estimate parameters $\alpha, \delta, \rho, \phi_1, \phi_2, \phi_3, \chi, \kappa$ with Bayesian estimation. For estimation, we consider the following equations.

$$\tilde{\imath}_t = \chi \tilde{\imath}_{t-1} + (1-\chi)\{\phi_1 E_t \tilde{\pi}_{t+1} + \phi_2 \tilde{g}_t\}, 0 < \chi < 1, 0 < \phi_1, 0 < \phi_2. \tag{24}$$

$$\hat{c}_t = \tilde{g}_t + \hat{c}_{t+1} - E_t \tilde{\imath}_{t+1} + E_t \tilde{\pi}_{t+1}. \tag{25}$$

$$E_t \hat{q}_{t+1} = \frac{1}{1-\delta}\left(\frac{1+i}{1+\pi}\left(\hat{q}_t + E_t(\tilde{\imath}_{t+1} - \tilde{\pi}_{t+1})\right) - \frac{r}{q}\hat{r}_t\right) \tag{26}$$

$$\widehat{w}_t = \kappa \hat{l}_t + \hat{c}_t \tag{27}$$

$$\hat{I}_t = \frac{1+i}{2+i+\pi}\hat{I}_{t-1} + \frac{1+i}{2+i+\pi}E_t \hat{I}_{t+1} + \frac{1+i}{(2+i+\pi)S''(1)}\hat{q}_t \tag{28}$$

$$\tilde{g}_t = \frac{I}{K}\hat{I}_t \tag{29}$$

$$\tilde{\pi}_t = E_t \tilde{\pi}_{t+1} + \frac{\rho^2}{1-\rho}\widehat{\omega}_t. \tag{32}$$

$$0 = \frac{c/K}{Y/K}\hat{c}_t + \frac{I/K}{Y/K}\hat{I}_t, or\ 0 = \frac{c}{K}\hat{c}_t + \frac{I}{K}\hat{I}_t. \tag{33}$$

$$\hat{a}_t = \phi_3 \hat{a}_t + f, 0 < \phi_3 < 1. \tag{34}$$

$$\widehat{w}_t = \widehat{\omega}_t + (1-\alpha)\hat{a}_t - \hat{l}_t, \tag{35}$$

$$\hat{r}_t = \widehat{\omega}_t + (1-\alpha)\hat{a}_t. \tag{36}$$

The parameters are given by the following table.

I/K	0.3
C/K	0.7

Table 1 Parameter setting

Growth rates of gross domestic product (GDP), consumption, nominal interest rate, inflation rate, increase rate of wage, and the unemployment rate are included in the model. The nominal interest rate and inflation rate are used to derive the real interest rate. Labor supply is regarded as the employment rate (1-unemployment rate). Data are annual data of 1995–2019 of the Cabinet Office, Japan. The parameter setting of prior estimation is shown as the following table. Data of the unemployment rate and real interest rate are subtracted using an HP filter.

	Distribution	Mean	Variance
α	uniform_pdf	0.3	0.1
δ	uniform_pdf	0.06	0.1
ρ	normal_pdf	0.25	0.2
ϕ_1	normal_pdf	2	4
ϕ_2	normal_pdf	0.2	0.25
ϕ_3	normal_pdf	0.9	0.5
χ	beta_pdf	0.5	0.25
κ	normal_pdf	2	4
stderr ea	inv_gamma_pdf	1.5	4
stderr ug	inv_gamma_pdf	1.5	4
stderr uc	inv_gamma_pdf	1.5	4
stderr ur	inv_gamma_pdf	1.5	4
stderr uw	inv_gamma_pdf	1.5	4
stderr ul	inv_gamma_pdf	1.5	4

Table 2. Prior Parameter Setting

Estimation can be reduced as the following table.

	Mean	Confidential Interval
α	0.32	0.3080 0.3364
δ	0.1659	0.1587 0.1727
ρ	0.5425	0.5381 0.5384
ϕ_1	0.65	0.3416 0.9457
ϕ_2	0.702	0.0175 1.3783
ϕ_3	0.3883	0.2654 0.4887
χ	0.7785	0.6047 0.9691
κ	1.3434	0.6906 1.9966
stderr ea	0.694	0.2562 1.1257
stderr ug	0.2401	0.2090 0.2620
stderr uc	0.3009	0.2639 0.3238
stderr ur	0.5345	0.4143 0.6703
stderr uw	0.3916	0.3544 0.4357
stderr ul	0.3576	0.3135 0.4042

Table 3. Posterior Parameter Setting

5. Simulation

In this section, we examine how the shock of the productivity a_t affects the macroeconomic variances. The equations in the simulation are presented below.

$$\tilde{\iota}_t = \chi\tilde{\iota}_{t-1} + (1-\chi)\{\phi_1 E_t\tilde{\pi}_{t+1} + \phi_2\tilde{g}_t\}, 0 < \chi < 1, 0 < \phi_1, 0 < \phi_2. \tag{24}$$

$$\hat{c}_t = \tilde{g}_t + \hat{c}_{t+1} - E_t\tilde{\iota}_{t+1} + E_t\tilde{\pi}_{t+1}. \tag{25}$$

$$E_t\hat{q}_{t+1} = \frac{1}{1-\delta}\left(\frac{1+i}{1+\pi}\left(\hat{q}_t + E_t(\tilde{\iota}_{t+1} - \tilde{\pi}_{t+1})\right) - \frac{r}{q}\hat{r}_t\right) \tag{26}$$

$$\widehat{w}_t = \kappa\hat{l}_t + \hat{c}_t \tag{27}$$

$$\hat{I}_t = \frac{1+i}{2+i+\pi}\hat{I}_{t-1} + \frac{1+i}{2+i+\pi}E_t\hat{I}_{t+1} + \frac{1+i}{(2+i+\pi)S''(1)}\hat{q}_t \tag{28}$$

$$\tilde{g}_t = \frac{I}{K}\hat{I}_t \tag{29}$$

$$\tilde{\pi}_t = E_t\tilde{\pi}_{t+1} + \frac{\rho^2}{1-\rho}\widehat{\omega}_t. \tag{32}$$

$$0 = \frac{c/K}{Y/K}\hat{c}_t + \frac{I/K}{Y/K}\hat{I}_t, or\ 0 = \frac{c}{K}\hat{c}_t + \frac{I}{K}\hat{I}_t. \tag{33}$$

$$\hat{a}_t = \phi_3\hat{a}_{t-1} + f, 0 < \phi_3 < 1. \tag{34}$$

$$\widehat{w}_t = \widehat{\omega}_t + (1-\alpha)\hat{a}_t - \hat{l}_t, \tag{35}$$

$$\hat{r}_t = \widehat{\omega}_t + (1-\alpha)\hat{a}_t. \tag{36}$$

The parameters are set as presented below. The parameters are important points for consideration.

α	0.32
δ	0.1659
ρ	0.5425
ϕ_1	0.65
ϕ_2	0.702
ϕ_3	0.3883
χ	0.7785
κ	1.3434
θ	1
I/K	0.3
C/K	0.7

Table 4 Parameter setting

The productivity shock effects are shown by the following figures.

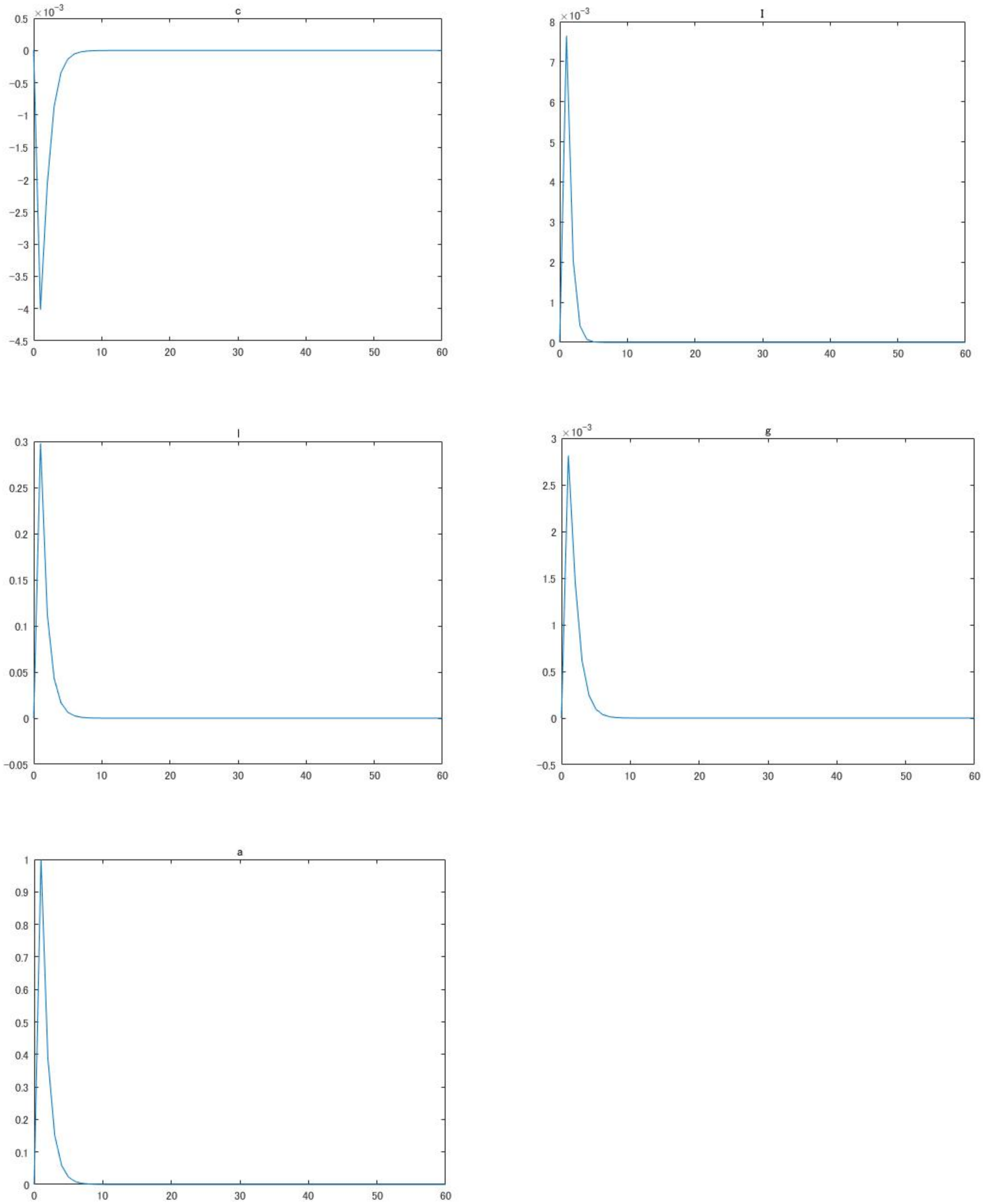

Fig. 1 Effects of Shock of Productivity.

An increase in the productivity a by 1% raises investment. This point is intuitive because the real interest rate increases. Then the return on investment increases. Because of increased investment, consumption decreases. An increase in the productivity raises the wage rate. Then the labor supply increases. The income growth rate g rises.

6. Conclusions

Our paper sets a DSGE model with an AK-type production function. Many DSGE-model-related papers include the assumption of a neoclassical product function. Considering the AK-type production function as an important component of the endogenous growth model, we can derive the equilibrium of the model economy as the balanced growth path. We can obtain the DSGE model with a simple AK-type product function as the contribution of our paper.[4]

[4] The derived equilibrium does not show the money stock. As long as monetary policy is considered, the setting of money stock is necessary. However, the derived results do not change even if the money stock is omitted. Our paper can show the general equilibrium model without consideration of the price setting, monetary policy or money.

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Program Code Shock of TFP

```
//1. variables
var c g i pi q r w l I a omega;

varexo f;

//2. parameter
parameters theta delta kappa IK rho CK kai phi1 phi2 phi3 alpha;

//2.1 parametervalue
theta=1;
rho=0.5425;
delta=0.1659;
kappa=1.3434;
IK=0.3;
CK=0.7;
kai=0.7785;
phi1=0.65;
phi2=0.702;
phi3=0.3883;
alpha=0.32;

//3.equations
model(linear);
c(-1)=g(-1)+c-1/theta*i+1/theta*pi;
q(+1)=1/(1-delta)*((1+0.01)/(1+0.01)*(q+(i(+1)-pi(+1)))-0.01/1*r);
w+(1-theta)*g=kappa*l+theta*c;
I=(1+0.01)/(2+0.01+0.01)*I(-1)+(1+0.01)/(2+0.01+0.01)*I(+1)+(1+0.01)/((2+0.01+0.01)*1/7)*q;
g=IK*I;
w=omega-l+(1-alpha)*a;
r=omega+(1-alpha)*a;
pi=pi(+1)+rho^2/(1-rho)*omega;
CK*c=-IK*I;
i=kai*i(-1)+(1-kai)*(phi1*pi(+1)+phi2*g);
a=phi3*a(-1)+f;
end;



//steady state check
steady;
```

```
check;

//5. simulation
shocks;
var f;
periods 1;
values 0.01;
end;

//6. results
simul(periods=60);

g1=g*100;
l1=l*100;
c1=c*100;
a1=a*100;
I1=I*100;

figure(1)
plot(0:60, g1(1:61)); title('g')
figure(2)
plot(0:60, l1(1:61)); title('l')
figure(3)
plot(0:60, c1(1:61)); title('c')
figure(4)
plot(0:60, a1(1:61)); title('a')
figure(5)
plot(0:60, I1(1:61)); title('I')
```

Program Code Calibration

```
var c g i pi q r w l I a omega g_obs c_obs r_obs w_obs l_obs;

varexo ug uc ur uw ul ea;

parameters    delta kappa IK rho CK kai phi1 phi2 phi3 alpha;
//2.1 parametervalue
//theta=1;
delta=0.06;
rho=0.25;
IK=0.3;
CK=0.7;
kai=0.9;
phi1=0.1;
phi2=0.1;
phi3=0.1;
alpha=0.3;
kappa=2;

model(linear);
c(-1)=g(-1)+c-i+pi;
q(+1)=1/(1-delta)*((1+0.01)/(1+0.01)*(q+(i(+1)-pi(+1)))-0.01/1*r);
w+g=kappa*l+c;
I=(1+0.01)/(2+0.01+0.01)*I(-1)+(1+0.01)/(2+0.01+0.01)*I(+1)+(1+0.01)/((2+0.01+0.01)*1/7)*q;
g=IK*I;
w=omega-l+(1-alpha)*a;
r=omega+(1-alpha)*a;
pi=pi(+1)+rho^2/(1-rho)*omega;
CK*c=-IK*I;
i=kai*i(-1)+(1-kai)*(phi1*pi(+1)+phi2*g);
a=phi3*a(-1)+ea;

g_obs=g+ug;
c_obs=c+uc;
r_obs=r+ur;
w_obs=w+uw;
l_obs=l+ul;
end;

//estimated_params;
```

```
// Setting Priors
estimated_params;
// START VALUES    LOWER BOUND    UPPER BOUND        PDF TYPE       PRIOR MEAN    STD. ERR.
delta,          0.06,              0,            ,        uniform_pdf,       0.06,          0.1;
alpha,          0.3,               0,           1,        uniform_pdf,       0.3,           0.1;
rho,            0.25,         0.0000,      0.9999,        normal_pdf,        0.25,          0.20;
phi1,           0.1,          0.0000,      10.000,        normal_pdf,        2.00,          4.00;
phi2,           0.1,          0.0000,      10.000,        normal_pdf,        0.20,          0.25;
phi3,           0.1,          0.0000,       1.000,        normal_pdf,        0.9,           0.5;
kai,            0.9,          0.0000,      0.9999,        beta_pdf,          0.50,          0.25;
kappa,          2.00,         0.0000,      10.000,        normal_pdf,        2.00,          4.00;

stderr ea,      1.50,         0.0000,      10.000,        inv_gamma_pdf,     1.50,          4.00;
stderr uc,      1.50,         0.0000,      10.000,        inv_gamma_pdf,     1.50,          4.00;
stderr ul,      1.50,         0.0000,      10.000,        inv_gamma_pdf,     1.50,          4.00;
stderr uw,      1.50,         0.0000,      10.000,        inv_gamma_pdf,     1.50,          4.00;
stderr ur,      1.50,         0.0000,      10.000,        inv_gamma_pdf,     1.50,          4.00;
stderr ug,      1.50,         0.0000,      10.000,        inv_gamma_pdf,     1.50,          4.00;

end;

varobs g_obs c_obs r_obs w_obs l_obs;

estimation(datafile = data, mode_check, mh_replic =500000, mh_nblocks =2, mh_drop =0.5, mh_jscale
=0.5, bayesian_irf, mcmc_jumping_covariance=identity_matrix);

//
```

Data file

```
data_q = [
0.018947172	-0.014729247	0.001389498	0.011	0.008139574
0.022966792	0.002242977	-0.002180579	0.011	0.006766645
0.013367525	0.016796825	-0.019964466	0.016	0.007375497
-0.013902419	0.004779282	-0.00816814	-0.013	0.000950247
-0.01754196	0.004204934	0.000214902	-0.015	-0.004529598
0.011898981	0.008510318	0.005196267	0.001	-0.004085124
-0.008402273	0.013086465	0.003787425	-0.016	-0.006734586
-0.014662569	0.007826148	0.006196603	-0.029	-0.010493685
-0.002132144	-0.000207662	0.000729656	-0.007	-0.009373916
0.009851863	0.003949424	-0.001911428	-0.005	-0.003380211
0.005239548	0.008394317	0.000974876	0.008	-0.000511647
0.004420462	0.0149802	-0.002608706	0.002	0.002234813
0.008234714	0.017950246	0.003239938	-0.009	0.003862526
-0.021817171	0.011571395	-0.010475446	-0.003	0.002373453
-0.062982778	-0.021555261	0.01654686	-0.038	-0.009232862
0.020473067	-0.015914264	0.009715124	0.006	-0.009958355
-0.019840758	-0.030238376	0.005927269	-0.003	-0.005799193
0.005019597	-0.020678266	0.003275149	-0.008	-0.003745319
0.013922535	-0.001793441	-0.000553087	-0.002	-0.001783052
0.018166341	0.00759731	-0.02297134	0.005	0.001103631
0.030237086	0.004639915	-0.003393161	0.001	0.001931866
0.004567822	-0.00780531	0.005682253	0.006	0.003718101
0.015697838	0.003578485	0.000457827	0.004	0.005477575
-0.001780715	0.006963239	-0.003667065	0.014	0.008223203
0.009002865	0.007222399	0.002206664	-0.005	0.006964479
];

g_obs = data_q(:,1);
c_obs = data_q(:,2);
r_obs = data_q(:,3);
w_obs = data_q(:,4);
l_obs = data_q(:,5);
```