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Information Projection and Timing Decisions: A Rationale for Second Thoughts

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Information Projection and Timing Decisions: A Rationale for Second Thoughts*

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Abstract

This study develops a dynamic model of information projection and explores how it affects timing of actions. An action is observable and available only after the agents' arrival. The value of an action is unknown; however, each agent receives a noisy signal on its value. Without information projection, if no one has taken action until then, the expected value of taking action reduces with the passage of time because inaction is a bad signal. In contrast, under projection bias, because of which the agent mistakenly believes that the other agent's arrival time is close to theirs, the expected action value may increase as time passes. Consequently, the agent has second thoughts; although they decide not to take action at the arrival timing, they overturn their initial decision and take action later.

Keywords: Delay, Information projection bias, Preemption games, Second thoughts, Social learning
JEL Classification: D81, D82, D83, D91

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1. Introduction

In daily observations, consumers often hesitate to purchase goods they find, even though they believe that they are worth purchasing. After deliberating whether to purchase goods, consumers might finally decide not to purchase them at that moment; however, we often observe that consumers continue to check the goods, care about other consumers' attitudes toward the goods, and eventually end up purchasing the goods. Although a standard model with unbiased agents can not explain the prevalence of such kind of second thoughts, this study provides a rationale for second thoughts in our dynamic model of decision timing by considering an established behavioral feature: *information projection bias* (Madarász, 2012).

We consider the situation with two (or more) agents; each happens to find a problem, and when found, the agent decides whether or not to announce the problem publicly, and announcing problem brings a private benefit for the first announcing agent. When the announcement is costly, the uncertainty about the benefit of announcing the problem makes it difficult for the agent to make the right decision. The agent must evaluate such uncertainty by observing the counterpart's behavior so far to make a better decision. More precisely, the agent finds the cutoff value, which indicates that the announcement's expected value overweighs its cost. In this situation, the agent updates their belief on the benefit of announcing the problem in the Bayes rule, which is based on the fact that no one has announced the problem yet. In general, the cutoff value increases over time, implying that if the agent decides not to announce the problem when found, they will not change their decision and will never announce it later; however, we observe that individuals have *second thoughts*. Although individuals are prone to making deliberate decisions at the beginning, they tend to change their initial decision later.

Considering the above situation in a dynamic model, we need to focus not only on the fact that the problem has not been announced but also the possible reasons of not announcing it. Moreover, when the counterpart has not announced the problem although they have already identified it, we ought to consider how the agent forms their beliefs about the timing when the counterpart initially acknowledged the problem. To consider this perspective, in addition to the standard Bayesian updating, we should suppose that the agents have information projection bias. Madarász (2012) introduces the notion of information projection bias and explains it as "people are aware of informational differences but project their private information, exaggerating the extent to which others have access to the content of their private information." In our context, the agent who finds the problem projects the information that they have about the problem to the counterpart's state about whether or not to have known the problem. Once the agent finds the problem, they suppose that the counterpart has known the problem since a particular time, which is likewise assumed by the agent.

The following two points should be carefully considered to formulate information projection bias in our model: (i) how the agent presumes the timing when the counterpart discovers the problem; (ii) such a presumption will either be fixed at the time when the agent finds the problem or will be updated every time after the agent finds the problem. Regarding the latter point, we examine two cases: the *fixed projection case* and *moving projection case*.

First, in the fixed projection case, we suppose that the agent projects their information when they find the problem, and this projection is fixed. Although the agent's belief update is affected by information projection bias, this impact only happens when the agent finds the problem. After that, the updating primarily follows the standard Bayesian rule. Under fixed projection, an equilibrium exists where the agent never announces the problem if they did not announce when they initially discovered it. We term this property *immediate announcement property* (IAP). This result holds regardless of the monotonicity of the cutoff value. Even if the cutoff value has reduced with the passage of time, it increases again when the agent finds the problem and projects it. As a result, fixed projection causes the equilibrium to have IAP.

Second, in the moving projection case, we suppose that the agent continuously projects their information over time after they find the problem. When the cutoff value is nondecreasing over time, an equilibrium with IAP exists; however, when the cutoff value is not monotone, the equilibrium no longer possesses IAP despite being present. When the agent's bias is strengthened over time, the cutoff value is non-monotonic, and an equilibrium exists with the property that although the agent decided not to announce the problem when they discovered it, they overturn their initial decision and announce it later. This result provides a rationale for second thoughts. Under fixed projection, the cutoff value increases again after finding the problem, even if it previously decreased. In contrast, the cutoff value can still decrease under moving projection even after finding the problem. This difference between the two projection types result in second thoughts under moving projection.

This paper initially contributes to the literature on information projection bias (Madarász, 2012, 2016). Madarász (2012) formalizes information projection bias in how agents project their private information and exaggerates how others know it. We extend the notion of information projection to our dynamic model and demonstrate that an agent has second thoughts when they continuously project information over time.[1] Furthermore, our study contributes to the literature on social learning.[2] Compared with ordinal observational learning models, such as Bikhchandani, Hirshleifer, and Welch (1992), our model is asymmetric in the sense that the decision on "not announcing" is unobservable and is related to Herrera and Hörner (2013). The authors show that no delay occurs in the decision using a model similar to the one used in this study, excluding information projection bias; however, we show that agents may have second thoughts if they are subject to information projection bias. Finally, our modeling is similar to that of preemption games (Brunnermeier and Morgan, 2010; Hopenhayn and Squintani, 2011; Bobtcheff, Bolte, and Mariotti, 2017) wherein we examine when a subject takes action by considering the decision made by counterparts who face the same situation. Moreover, most research on preemption games supposes that once a party takes action, others never have any

[1] Related to information projection bias, Gagnon-Bartsch (2016), Gagnon-Bartsch, Pagnozzi, and Rosato (2021), and Gagnon-Bartsch and Rosato (2022) study taste projection. Based on the formulation of taste projection by Gagnon-Bartsch (2016), Gagnon-Bartsch, Pagnozzi, and Rosato (2021) demonstrate how taste projection influences bitter behaviors in an auction, and Gagnon-Bartsch and Rosato (2022) study the effects of taste projection on consumer behavior and pricing in markets.

[2] Bikhchandani, Hirshleifer, Tamuz, and Welch (2021) is an excellent survey on this topic.

cost and benefit from the situation. Although following these settings, we focus on the agent's information projection bias and explain why individuals occasionally overturn their decisions earlier.

The rest of the paper is structured as follows. We introduce our model in the next section, formulate information projection bias, and define equilibrium. In Section 3, we examine an equilibrium under fixed projection and monotonicity of the cutoff strategy, which serves as a benchmark case for subsequent analysis under moving projection. Section 4 shows an equilibrium under moving projection and explains how information projection bias leads to the agent's second thoughts. We conclude by discussing a few issues of our model in Section 5.

2. Model

2.1. Setting

Time $t \in (\underline{T}, \overline{T})$, where $-\infty < \underline{T} < \overline{T} < \infty$, is continuous. Let N be the set of agents who will face a problem. Each agent finds the problem with probability $\lambda_t dt$ at each period, where $\lambda_t \in (0, M)$ for some $M \in \mathbb{R}_{++}$. We assume that $\Lambda(t) = \int_{\underline{T}}^{t} \lambda_s ds \in \mathbb{R}_{++}$ and $\lim_{t\to\infty} \frac{\Lambda(t)}{t} > 0$. Only when the agent finds the problem can they determine whether to announce it. Although the agent does not announce the problem when they find it, they can announce it later. Once an agent announces the problem, the game ends.

Agents cannot observe whether others find the problem, but they can observe the announcement made by someone who found it. Announcing the problem yields gain $V \in \{0, 1\}$ but incurs cost $k \in (0, 1)$. Agents who do not announce the problem gain payoff 0.[3] The prior belief about V is that $\Pr(V = 1) = 1/2$. The true value of V is unknown to all agents; however, each agent receives signal $\theta_i \in (\underline{\theta}, \overline{\theta}) \subsetneq \mathbb{R}_{++}$ when they find the problem. We assume that each agent's signal is independent and identically drawn. The probability density function of

[3]If some agents announce the problem simultaneously, a tie is broken in some arbitral manner.

θ under V realization is $f_V(\cdot)$ and its CDF is $F_V(\cdot)$. Each f_V has full support on $(\underline{\theta}, \overline{\theta})$. We assume monotone likelihood dominance; $\frac{f_1(\theta)}{f_0(\theta)}$ is monotonically increasing in θ. Therefore, larger θ implies the likelihood of the problem having high value. For the sake of simplicity, we further assume $\frac{f_1(\theta)}{f_0(\theta)} = \theta$.

Let $(a_{it})_{i\in N} \in \{1,0\}^{|N|}$ be the action profile for each $t \in (\underline{T}, \overline{T})$ and τ_i be the timing when agent i's finds the problem. Here $a_{it} = 1$ implies agent i announced the problem at t. Then, the private history of agent i at t is $h_{it} = ((a_{is})_{i\in N, s\leqslant t}, \varnothing)$ if $t < \tau_i$, and $h_{it} = ((a_{is})_{i\in N, s\leqslant t}, (\tau_i, \theta_i))$ if $t \geqslant \tau_i$. Now agent i's behavioral strategy is a function $\sigma_{it} : h_{it} \mapsto a_{it}$. Here, we assume that the agent cannot announce the problem at t (i) if they do not find it: $\sigma_{it}(h_{it}) = 0$ if $a_{is} = 1$ for each $t < \tau_i$, and (ii) if someone, including them, have already announced it: $\sigma_{it}(h_{it}) = 0$ if $a_{js} = 1$ for some $j \in N$ and $s < t$. We denote the strategy profile by σ.

2.2. Information Projection and Belief Formation

After receiving a signal, agents update their beliefs.[4] This corresponds to the standard Bayesian belief updating. Additionally, we suppose that the agent misperceives their counterpart's state or information, which, in turn, influences their formation of belief; when the agent finds the problem, they project the fact to the counterpart's state, and they believe that their counterpart must have already found the problem. This kind of bias affects the agent's belief updating referred to as information projection bias following Madarász (2012). Let α_t be the degree of the bias at t, which is supposed to be identical for all agents. The agent's belief on the value of the announcement (V) at t depends both on the bias with probability of α_t and on Bayesian updating with probability of $1-\alpha_t$. Suppose that the agent finding the problem at t believes that their counterpart has already found it at $\zeta(t) < t$. We assume that ζ is nondecreasing. In sum, the agent updates their belief on V based on the received signal and the counterpart's action profile under the condition where the agent believes that the counterpart found the problem at

[4]For the sake of simplicity, we suppose that $N = 2$. We can extend the following analysis to which there are N agents.

$\zeta(t)$.

Furthermore, let $\beta^{\tau}_{it}[\tau_i \mid \sigma]$ be the belief at t on τ_j under strategy σ when agent i finds the problem at τ_i and projects the information to the counterpart's state at τ. For each $t \geqslant \tau_i$, the belief β is defined as follows:

$$\beta^{\tau}_{it}[\tau_i \mid \sigma](s) = \begin{cases} \alpha_\tau + (1-\alpha_\tau)\Pr(\tau_j = s \mid h_{it}, \sigma) & \text{if} \quad s = \zeta(\tau), \\ (1-\alpha_\tau)\Pr(\tau_j = s \mid h_{it}, \sigma) & \text{if} \quad s \neq \zeta(\tau), \end{cases}$$

where $\Pr(\tau_j = s \mid h_{it}, \sigma)$ is the Bayesian belief observing history under strategy profile σ. When the timing of agent i presuming that the counterpart found the problem is consistent with the timing of the counterpart actually finding the problem ($\zeta(\tau) = \tau_j$), then the agent's belief concerning the bias is accurate. Then, this part of belief is $\alpha_\tau \cdot 1 = \alpha_\tau$ when the degree of the bias at τ is α_τ. The overall belief comprises this bias term and the standard Bayesian updating term with a rate of $(1-\alpha_\tau)$, represented by $(1-\alpha_\tau)\Pr(\tau_j = s \mid h_{it}, \sigma)$.[5] Conversely, when $\zeta(\tau) \neq \tau_j$, the agent's belief concerning the bias is incorrect, and the bias term is $\alpha_\tau \cdot 0 = 0$. As a result, the prevailing belief is represented only by the Bayesian term with a rate of $(1-\alpha_\tau)$.

We consider two kinds of information projection classified by the timing of projection: fixed and moving. Under fixed projection, the agent projects their information when they find the problem, and this projection is fixed after that; if agent i finds a problem at τ_i, in any period $t \geqslant \tau_i$, they presume that the counterpart found it at $\zeta(\tau_i) \geqslant \tau_i$. In contrast, under moving projection, the projection bias evolves even after the agent finds the problem; if agent i finds a problem at τ_i in period $t \geqslant \tau_i$, they presume that the counterpart found it at period $\zeta(t) \geqslant \tau_i$. In the definition of $\beta^{\tau}_{it}[\tau_i \mid \sigma](s)$, $\tau = \tau_i$ under fixed projection and $\tau = t$ under moving projection.

[5] We call the former term the *bias term* and the latter term the *Bayesian term*.

2.3. Definition of the equilibrium

The agents have the common discount rate denoted by ρ. Now we define the equilibrium.

Definition 1. Let $\varsigma_{it}^t = \inf\{s : \sigma_{is}^t(h_{is}) = 1 \text{ and } s > t\}$ be the stopping time if agent i decides not to announce at t under the belief of the agent at t. The strategy profile, σ, is an equilibrium if, for each t and each i,

$$\begin{aligned}
\sigma_{it}(h_{it}) = \arg\max_a\, & a(E[V \mid \beta_{it}^\tau, h_{it}, \sigma] - k) \\
& + (1-a)E_{\varsigma_{it}^t}[e^{-\rho(\varsigma_{it}^t - t)}\Pr(a_{js} = 0, s < \varsigma_{it} \mid \beta_{it}^\tau, h_{it}, \sigma)(E[V \mid \beta_{i\varsigma_{it}^t}^\tau, h_{i\varsigma_{it}^t}, \sigma] - k)], \\
\sigma_{is}^t(h_{is}) = \arg\max_a\, & a(E[V \mid \beta_{is}^\tau, h_{is}, \sigma] - k) \\
& + (1-a)E_{\varsigma_{is}^t}[e^{-\rho(\varsigma_{is}^t - s)}\Pr(a_{js'} = 0, s' < \varsigma_{is} \mid \beta_{is}^\tau, h_{is}, \sigma)(E[V \mid \beta_{i\varsigma_{is}^t}^\tau, h_{i\varsigma_{is}^t}, \sigma] - k)]
\end{aligned}$$

where $\tau = \tau_i$ under fixed projection and $\tau = t$ under moving projection.

In this definition, agents behave rationally despite having information projection bias. The bias is common knowledge for all agents. Note that under fixed and moving projection, each agent anticipates that their future self has the same bias at the decision period. Indeed, at period t, the agent in future periods ($s > t$) plays σ_{is}^t, which maximizes the payoff under the belief β_{is}^τ. By definition, the bias terms of β^τ depends on $\tau \in \{\tau_i, t\}$. In this sense, agents are naïve about evolution of future biases.[6]

We make the following assumption, guaranteeing the existence of equilibrium.[7]

Assumption 1. (i) There is $t^* > \underline{T}$ such that for each $t \leqslant t^*$, $\zeta(t) = \underline{T}$. (ii) $1 < \frac{1}{\underline{\theta}}\frac{k}{1-k} < \overline{\theta}$.

$1 < \frac{1}{\underline{\theta}}\frac{k}{1-k}$ implies that $k > \frac{\underline{\theta}}{1+\underline{\theta}} = \frac{f_1(\underline{\theta})}{f_1(\underline{\theta})+f_0(\underline{\theta})}$. That is, when a Bayesian agent receives $\theta = \underline{\theta}$, they never desire to announce the problem. $\frac{1}{\underline{\theta}}\frac{k}{1-k} < \overline{\theta}$ implies that $\frac{f_1(\overline{\theta})f_1(\underline{\theta})}{f_1(\overline{\theta})f_1(\underline{\theta})+f_0(\overline{\theta})f_0(\underline{\theta})} > k$. This

[6] This distinction does not matter in fixed projection; however, it plays an important role for the equilibrium characterization in moving projection. Section 5 discusses how our results alter if agents rationally anticipate the evolution of future biases.

[7] Appendix C provides the proof for the existence of the equilibrium.

implies that even when the agent is aware that the counterpart receives $\underline{\theta}$, they are willing to announce the problem if they receive $\overline{\theta}$.

This paper focuses on the symmetric equilibrium. Let θ_{t,τ,τ_i} be the cutoff at t regarding whether to announce the problem when agent i projects their information at $\tau \geqslant \tau_i$. Moreover, let $(\theta_{t,\tau,\tau_i})_{t\in(\underline{T},\overline{T})}$ be a sequence of cutoffs. Then, we consider the strategy with $(\theta_{t,\tau,\tau_i})_{t\in(\underline{T},\overline{T})}$: $a_{it} = 1$ if $t \geqslant \tau_i$, $\theta_i > \theta_{t,\tau,\tau_i}$, and $a_{jt'} = 0$ for each $j \in N$ and $t' < t$; otherwise $a_{it} = 0$.We term this strategy *the cutoff strategy* $(\theta_{t,\tau,\tau_i})_{t\in(\underline{T},\overline{T})}$. Note that $(\theta_{t,\tau,\tau_i})_{t\in(\underline{T},\overline{T})} = (\theta_{t,\tau_i,\tau_i})_{t\in(\underline{T},\overline{T})}$ under fixed projection and $(\theta_{t,\tau,\tau_i})_{t\in(\underline{T},\overline{T})} = (\theta_{t,t,\tau_i})_{t\in(\underline{T},\overline{T})}$ under moving projection.[8]

In subsequent sections, we find equilibria with the cutoff strategy (θ_{t,τ,τ_i}) in fixed and moving projection cases and characterize the strategies and the agent's announcement behavior on the equilibrium path.

3. Equilibrium under Fixed Projection

This section examines equilibria under fixed projection. As described, under fixed projection, the agent projects their information when they find the problem; this projection is fixed following that. We show that the equilibrium under fixed projection has *immediate announcement property* (IAP). Formally, the equilibrium has IAP: for each i, t, and τ_i, $a_{it} = 0$ for each $t > \tau_i$ on the equilibrium path when $a_{i\tau_i} = 0$.

Under fixed projection, the condition that agent i announces the problem at t is provided by $P_{t,\tau_i}(\theta_i) - k \geqslant 0$ where $P_{t,\tau_i}(\theta_i)$ is the probability that $V = 1$ under agent i's belief at period t. We demonstrate a cutoff sequence $(\hat{x}_t)$ so that the condition can be reformulated as $\theta_i \geqslant \hat{x}_t$. To observe this, we must calculate $P_{t,\tau_i}(\theta_i)$. Let Q_t^V be the probability that the agent has announced the problem until t under state $V \in \{0, 1\}$. Under the cutoff strategy, we can calculate Q_t^V as follows:[9]

[8] For the sake of notational simplicity, we will omit the subscript of $(\theta_{t,\tau,\tau_i})_{t\in(\underline{T},\overline{T})}$ and represent it as (θ_{t,τ,τ_i}) when it is obvious.

[9] All proofs are provided in Appendix A.

Lemma 1. *If the cutoff strategy with* $(\hat{x}_t)$ *is at an equilibrium state that has IAP,*

$$Q_t^V = \int_{\underline{T}}^{t} e^{-\Lambda(t')}\lambda_{t'}(1 - F_V(\hat{x}_{t'}))dt'.$$

$P_{t,\tau_i}(\theta_i)$ can be represented by

$$P_{t,\tau_i}(\theta_i) := \frac{\theta_i(\alpha_{\tau_i}F_1(\hat{x}_{\zeta(\tau_i)}) + (1-\alpha_{\tau_i})(1-Q_t^1))}{\theta_i(\alpha_{\tau_i}F_1(\hat{x}_{\zeta(\tau_i)}) + (1-\alpha_{\tau_i})(1-Q_t^1)) + (\alpha F_0(\hat{x}_{\zeta(\tau_i)}) + (1-\alpha_{\tau_i})(1-Q_t^0))}.$$

Then, announcing at t is optimal if and only if $P_{t,\tau_i}(\theta_i) - k \geqslant 0$, equivalently, $\theta \geqslant \hat{x}_{t,\tau_i}$, where

$$\hat{x}_t = \frac{\alpha_t F_0(\hat{x}_{\zeta(t)}) + (1-\alpha_t)(1-Q_t^0)}{\alpha_t F_1(\hat{x}_{\zeta(t)}) + (1-\alpha_t)(1-Q_t^1)}\frac{k}{1-k}. \tag{1}$$

As defining

$$\hat{x}_{t,\tau_i} := \frac{\alpha_{\tau_i} F_0(\hat{x}_{\zeta(\tau_i)}) + (1-\alpha_{\tau_i})(1-Q_t^0)}{\alpha_{\tau_i} F_1(\hat{x}_{\zeta(\tau_i)}) + (1-\alpha_{\tau_i})(1-Q_t^1)}\frac{k}{1-k},$$

we have the following proposition:

Proposition 1. *Suppose that sequence* $(\hat{x}_t)$ *satisfies Equation* (1). *Under fixed projection, there is a cutoff equilibrium having IAP, and the cutoff satisfies* $\theta_{t,\tau_i,\tau_i} = \hat{x}_{t,\tau_i}$.

The intuition is as follows. Under fixed projection, once the agent finds a problem, the timing by when the agent projects their information is fixed; Bayesian updating primarily affects the agent's belief later. As time passes, the probability that the counterpart has already found the problem increases. This implies a negative signal on the value of the announcement when the counterpart does not announce the problem. This information externality is more likely to discourage the incentive to announce the problem over time. As a result, under fixed projection, the agent never announces the problem if they decide not to do so when they discover it.

The following proposition provides sufficient conditions for the monotonicity of $\hat{x}_t$.

Proposition 2. *(i) If $\zeta(t) = \underline{T}$ for each $t \in [\underline{T}, \overline{T}]$ and $\dot{\alpha}_t \geqslant 0$ for each $t \in [\underline{T}, \overline{T}]$, $\hat{x}_t$ is strictly increasing in t. (ii) If $\alpha_t = 0$ for each $t \in [\underline{T}, \overline{T}]$, $\hat{x}_t$ is strictly increasing in t. (iii) If $\alpha_t = 1$ for each $t \in [\underline{T}, \overline{T}]$, $\hat{x}_t$ is constant.*

(i) represents the case where the agent who finds the problem believes that their counterpart has already found it in the beginning, and their bias strengthens as time passes, and (ii) can be witnessed as the benchmark case where the agent does not possess information projection bias. In this case, $\hat{x}_t$ is strictly increasing, implying that the later the agent finds the problem, the less likely they will announce it. This comes from the Bayesian term: the counterpart is more likely to find the problem as time passes. No announcement implies that the received signal is weak; thus, the agent infers from this. In contrast, (iii) is the case where the agent completely projects their information to their counterparts. In this case, the agent believes that their counterpart found the problem in a specific period, namely, $\zeta(t)$; however, the decision in $\zeta(t)$ is also based on the decision in $\zeta(\zeta(t))$. Repeating this process implies that the decision is based on $\zeta(\zeta(\cdots\zeta(t)\cdots)) = \underline{T}$. Therefore, the cutoffs are constant.

In contrast, there are some conditions where $(\hat{x}_t)$ is not monotonic. To observe this, we make the following assumptions:

Assumption 2. (a) Condition for α: $\dot{\alpha}_t > 0$ for each t and $\lim_{t\to\infty} \frac{e^{-\Lambda(t)}\lambda_t}{\dot{\alpha}_t} = 0$.

(b) Condition for ζ: (i) For each $t > t^*$, $\zeta(t)$ is strictly increasing in t, and (ii) $\lim_{\overline{T}\to\infty} \lim_{t\to\overline{T}} te^{-\Lambda(\zeta(t))} = 0$.

Assumption 3. (a) Condition for α: $\dot{\alpha}_t < 0$ for each t and $\lim_{t\to\infty} \frac{e^{-\Lambda(t)}\lambda_t}{\dot{\alpha}_t} = 0$.

(b) Condition for ζ: $\zeta(t) = \underline{T}$ for each t.

Assumption 2 represents where the agent's bias has strengthened over time; the agent believes that the later the agent finds the problem, the closer is the timing of their counterpart finding it.

In contrast, Assumption 3 represents a situation where the agent's bias has weakened over time; the agent believes that their counterpart found the problem initially. Under Assumptions 1 to 3, we have the following proposition on the nonmonotonicity of $\hat{x}_t$:

Proposition 3. *Under Assumption 1 and either Assumption 2 or Assumption 3, if $\overline{T}$ is sufficiently large, there are t, t' with $t' > t$ such that $\hat{x}_t > \hat{x}_{t'}$.*

The intuition for these results is as follows. The cutoff value of agent i is higher if the inferred θ_j from agent $j \neq i$'s silence is low. Under Assumptions 1 and 2, suppose by contradiction that $\hat{x}_t$ increases. Then, as t increases, even if agent i does not announce at period t, the inferred θ_i from the event becomes higher. As $\zeta(t)$ increases, in the biased term, the counterpart of i believes that the inferred θ_i increases. The cutoff value can decrease if this effect overcomes the Bayesian term's effect. Conversely, under Assumptions 1 and 3, the projected period is $\underline{T}$. If $\hat{x}_t$ increases, inferred θ_i from the no announcement by the agent in period $\underline{T}$ is the lowest. Then, as the projection bias weakens, this effect decreases as time passes.

4. Moving Projection and Second Thoughts

4.1. Equilibrium under Moving Projection

This section investigates equilibria under moving projection. As described, under moving projection, the agent updates the bias term on their belief and the Bayesian term every time after finding the problem. This implies that a sequence of cutoffs in equilibrium is possibly independent of the timing when the agent finds the problem. Let $\tilde{P}_t(\theta_i)$ be the probability that $V = 1$ under agent i's belief at period t. When the agents are sufficiently myopic, the condition that agent i announces the problem at t is $\tilde{P}_t(\theta_i) \geqslant k$. As in the fixed projection case, we show a cutoff sequence $(\tilde{x}_t)$ so that the condition can be reformulated as follows: $\theta_i \geqslant \tilde{x}_t$.

Let $\tilde{Q}_t^V$ be the probability under agent j's belief that agent $i \neq j$ has announced the problem until t under the state $V \in \{0, 1\}$. As in Lemma 1, under the cutoff strategy, we can calculate

$\tilde{Q}_t^V$ as follows:

Lemma 2. *If the cutoff strategy with $(\tilde{x}_t)$ is an equilibrium,*

$$\tilde{Q}_t^V = \int_{\underline{T}}^{t} \underbrace{e^{-\Lambda(t')}\lambda_{t'}}_{\text{prob. finds at } t'} (1 - F_V(\xi_t(t'))dt',$$

where $\xi_t(t') = \min_{t'' \in [t',t]} \tilde{x}_{t''}$.

ξ implies that the decision can be delayed: if $\tilde{x}_t$ is decreasing at some t, i.e., $\tilde{x}_{t'} > \theta > \tilde{x}_t$ for some $t > t'$, the agent who finds a problem at t' and receives θ will announce the problem at period t although they decide not to announce it at the period t'. Under moving projection, $\tilde{P}_t(\theta_i)$ can be represented as follows:

$$\tilde{P}_t(\theta_i) := \frac{\theta_i(\alpha_t F_1(\xi_t(\zeta(t)) + (1-\alpha_t)(1-\tilde{Q}_t^1))}{\theta_i(\alpha_t F_1(\xi_t(\zeta(t)) + (1-\alpha_t)(1-\tilde{Q}_t^1)) + (\alpha_t F_0(\xi_t(\zeta(t)) + (1-\alpha_t)(1-\tilde{Q}_t^0))}.$$

Then, if $\rho = \infty$, announcing at t is optimal if and only if $\tilde{P}_t(\theta) - k \geqslant 0$, equivalently, $\theta \geqslant \tilde{x}_t$, where

$$\tilde{x}_t = \frac{\alpha_t F_0(\xi_t(\zeta(t))) + (1-\alpha_t)(1-\tilde{Q}_t^0)}{\alpha_t F_1(\xi_t(\zeta(t))) + (1-\alpha_t)(1-\tilde{Q}_t^1)} \frac{k}{1-k}. \tag{2}$$

We have the following proposition:

Proposition 4. *Suppose a sequence $(\tilde{x}_t)$ satisfies (2). Under moving projection, there is a cutoff equilibrium that satisfies $\theta_{t,t,\tau_i} = \tilde{x}_t$ for each $\tau_i \leqslant t$.*

This proposition implies that the agent announces the problem as soon as $\theta > \tilde{x}_t$. If $\tilde{x}_t$ is nondecreasing, once the agent decides not to announce the problem, they will never announce it later. This is because if $\tilde{x}_t$ is nondecreasing, $\tilde{x}_{t'} > \theta$ holds for each $t' > t$. In contrast, if $\tilde{x}_t$ is nonmonotone, there is a possibility of *second thoughts*; although the agent decided not to announce the problem when they found it, they overturn their initial decision and announces it later. That is, $\tilde{x}_t > \theta > \tilde{x}'_t$ for some t and $t' > t$.

4.2. Second Thoughts under Moving Projection

We study equilibrium properties under moving projection by comparing them with those under fixed projection. We provide a rationale for second thoughts caused by moving projection.

We first examine the differences between $\tilde{x}_t$ and $\hat{x}_t$, previously defined, as follows:

$$\tilde{x}_t = \frac{\alpha_t F_0(\xi_t(\zeta(t))) + (1-\alpha_t)(1-\tilde{Q}_t^0)}{\alpha_t F_1(\xi_t(\zeta(t))) + (1-\alpha_t)(1-\tilde{Q}_t^1)} \frac{k}{1-k}$$
$$\hat{x}_t = \frac{\alpha_t F_0(\hat{x}_{\zeta(t)}) + (1-\alpha_t)(1-Q_t^0)}{\alpha_t F_1(\hat{x}_{\zeta(t)}) + (1-\alpha_t)(1-Q_t^1)} \frac{k}{1-k}.$$

As $\xi_t(s) = \min_{t' \in [s,t]} \tilde{x}_{t'}$, $\xi_t(s) = \tilde{x}_s$ if $\tilde{x}_t$ increases in t. Therefore, if $\tilde{x}_t$ is nondecreasing, $\tilde{x}_t = \hat{x}_t$. This also implies that if $\hat{x}_t$ has decreasing points, $\tilde{x}_t$ is nonmonotone. Therefore, if $(\hat{x}_t)$ is monotonic, there is no difference in fixed and moving projection. In other words, differences between moving and fixed projection arise when $(\hat{x}_t)$ is nonmonotone. The equilibrium under fixed projection has IPA even when the cutoff sequence is nonmonotone; however, this result cannot hold under moving projection. The agent may announce the problem after deciding not to do so when they find it; this is summarized in the following proposition.

Proposition 5. *(a) If there is a nondecreasing sequence $(\hat{x}_t)$ satisfying (1), an equilibrium exists with the cutoff strategy $(\tilde{x}_t) = (\hat{x}_t)$ that has IAP under moving projection.*

(b) If any sequence $(\hat{x}_t)$ satisfying (1) is nonmonotonic, no equilibrium exists with the cutoff strategy $(\tilde{x}_t)$ that has IAP under moving projection.

When the agent finds the problem in the region where the cutoff sequence decreases, the difference between moving and fixed projections impacts the agents' decision of the announcement later. As in Figure 1, the cutoff sequence decreases after finding the problem under

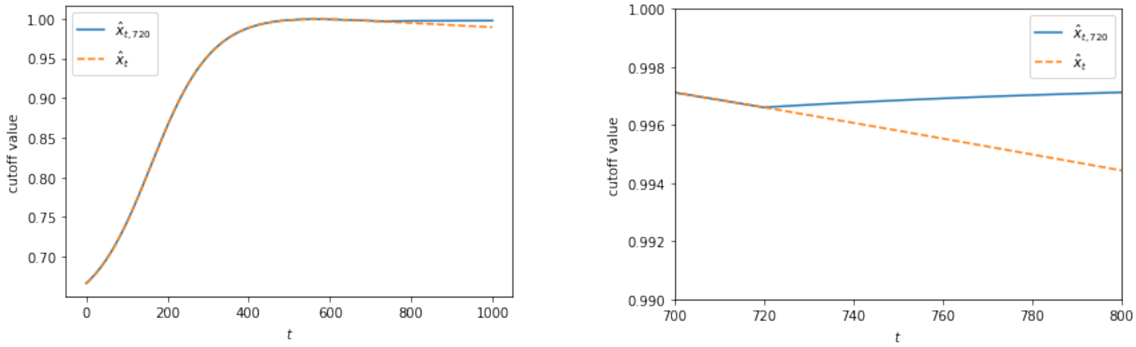

Figure 1.: Nonmonotonic $(\hat{x}_t)$ and $(\hat{x}_{t,\tau})$

moving projection although it increases under fixed projection. This property of moving projection provides a rationale for second thoughts.

Theorem 1 (Second thought)**.** *Under the cutoff strategy with* $(\tilde{x}_t)$*, if* $(\tilde{x}_t)$ *is nonmonotonic, there are* t, t' *with* $t' > t \geqslant \tau_i$*; an agent decides not to announce the problem at period* t *but decides to announce it at* t'*.*

When the agent does not have information projection bias, they predict that the benefit of the announcement would be low if their counterpart has not announced it. This increases the cutoff sequence as time passes, and second thoughts do not occur. In contrast, this prediction may not always hold true when the agent has information projection bias. Under fixed projection, the agent updates their belief depending only on the Bayesian term after discovering the problem because their information projection is fixed. Then, the cutoff sequence increases again after finding the problem despite initially decreasing, and the agent never overturns the initial decision. In contrast, under moving projection, because the agent projects their information every time after finding the problem, the bias and Bayesian terms continue to affect their belief update. Consequently, the cutoff sequence decreases even after finding the problem, and this causes the possibility of second thoughts such that the agent announces the problem later although they did not announce it when it was discovered.

To confirm the result above, suppose by contradiction the cutoff sequence increases after finding the problem. Under Assumption 2, if the agent finds the problem at a sufficiently later period, they believe that the counterpart did so as well. As the cutoff sequence increases, even when the counterpart receives a clear signal for $V = 1$, the counterpart may not make an announcement. Therefore, no announcement does not imply that the counterpart receives a weak signal. Then, the inference from no announcement becomes weaker, and as a result, the cutoff sequence continues to decrease even after finding the problem. In this case, under fixed projection, the agent never updates their bias, i.e., even at a later period, the belief about the timing of the counterpart's finding does not change. Therefore, the agent never overturns their initial decision. In contrast, under moving projection, the agent updates their bias as time passes, which causes them to believe that the counterpart will find the project at a later time; thus, their initial decision may be overturned. As a result, second thoughts can be explained when the agent has information projection bias under moving projection.[10]

5. Discussions and Conclusion

We considered agents with information projection bias and studied the timing when they announced the problem they found. When the agent projects information to the counterpart's state every time after finding the problem, the cutoff value representing the condition of announcing the problem potentially decreases. This causes the agent to have second thoughts by announcing the problem later, even if they did not do so when they initially found the problem.

We discuss a few concerns of our model.[11] First, we provide a comparative analysis on α. As in Figure 2, compared with the full bias case ($\alpha = 1$), the cutoff value for the agent with a partial bias($\alpha \in (0, 1)$) was relatively low at earlier stages but increased in later stages. We

[10] Second thoughts can be observed only when the cutoff sequence is decreasing, which happens under Assumption 3 as well as Assumption 2. Under Assumption 3, the agent thinks that their counterpart has been aware of the problem since the beginning. Because such a counterpart makes an announcement even when the signal is weak, the counterpart not announcing the problem is a strong signal of $V = 0$; however, the bias effect decreases over time and the cutoff sequence lessens.

[11] Formal propositions related to the following discussion are provided in Appendix B.

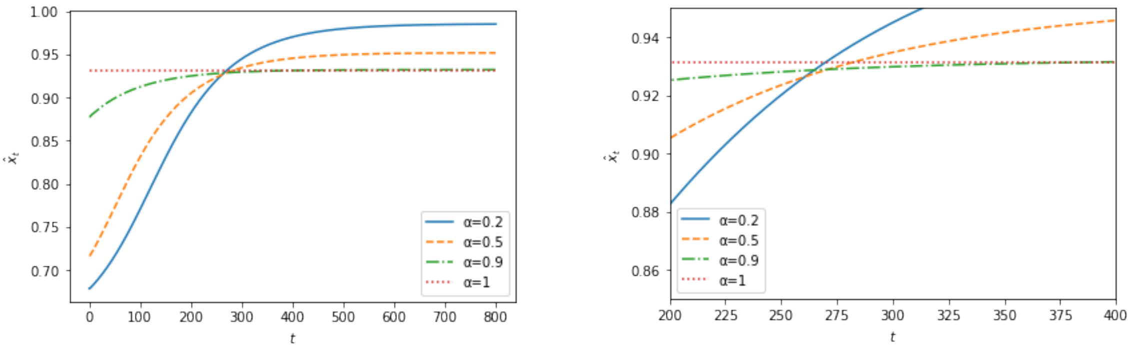

Figure 2.: Comparative statics of α

can infer from Proposition 2 that only the partially biased agents may have second thoughts. Second, the equilibrium under moving projection depends on the assumption that each agent is naïve about their future bias. If the agent correctly anticipates their future bias, the cutoff equilibrium shown in Proposition 4 cannot hold. The agent may deviate and announce the problem immediately after finding it, even if it is costly because such an announcement prevents them from having second thoughts in the future.

In our model, the cost and benefit of announcing the problem affect only the payoff for the agent who announces it; however, in some situations, the effects of announcement are interdependent among agents. For example, if one of the board members finds a problem with the company and announces it publicly, such an announcement can affect the entire company's financial situation, including other board members. We could study this situation by extending our model to consider the announcement as a public good. This would be one of our future research topics.

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A. Proofs

Before proceeding to the proofs, we provide the following lemma.

Lemma 3. *(a) Define* $\kappa_t = \frac{\alpha_t F_1(x_{\zeta(t)})}{\alpha_t F_1(x_{\zeta(t)})+(1-\alpha_t)(1-Q_t^1)} \in (0,1)$*. Then,*

$$\hat{x}_t = \left(\kappa_t \frac{F_0(x_{\zeta(t)})}{F_1(x_{\zeta(t)})} + (1-\kappa_t)\frac{1-Q_t^0}{1-Q_t^1}\right)\frac{k}{1-k}.$$

(b) $\frac{1-Q_t^0}{1-Q_t^1}$ *is a convex combination of* $\{\frac{F_0(x_{t'})}{F_1(x_{t'})}\}_{t'<t} \cup \{1\}$*.*

(c) MLHP implies $\frac{f_0(x)}{f_1(x)} < \frac{F_0(x)}{F_1(x)}$ *for each* x*. Then,* $\frac{F_0(x)}{F_1(x)}$ *decreases in* x*.*

(d) For each t*,* $\hat{x}_t \in [\frac{k}{1-k}, \frac{1}{\underline{\theta}}\frac{k}{1-k}]$*.*

(e) Under Assumption 1 (b), $0 < F_V(\hat{x}_t) < 1$*, and* $Q_t^V \in (0,1)$*.*

Proof of Lemma 3. (a) is immediately followed from (1).

(b) Note that $\frac{1-Q_t^0}{1-Q_t^1}$ is rewritten as follows:

$$\frac{1-Q_t^0}{1-Q_t^1} = \frac{1-\int_{\underline{T}}^t e^{-\Lambda(t')}\lambda_{t'}dt'}{1-Q_t^1} + \int_{\underline{T}}^t \frac{e^{-\Lambda(t')}\lambda_{t'}dt' F_1(\hat{x}_{t'})}{1-Q_t^1}\frac{F_0(\hat{x}_{t'})}{F_1(\hat{x}_{t'})}.$$

Note that by the definition of Q_t^1,

$$\frac{1-\int_{\underline{T}}^t e^{-\Lambda(t')}\lambda_{t'}dt' + \int_{\underline{T}}^t e^{-\Lambda(t')}\lambda_{t'}dt' F_1(\hat{x}_{t'})}{1-Q_t^1} = 1.$$

Thus, $\frac{1-Q_t^0}{1-Q_t^1}$ is a convex combination of $\left\{\frac{F_0(\hat{x}_{t'})}{F_1(\hat{x}_{t'})}\right\}_{t'<t} \cup \{1\}$.

(c) As MLHP order implies the reverse hazard ratio order, the definition of the reverse hazard ratio order implies the inequality (e.g., Shaked and Shanthikumar, 2007). A differentiation shows that $\frac{F_0(x)}{F_1(x)}$ is decreasing in x.

(d) As $\frac{F_0(x)}{F_1(x)}$ is decreasing in x by (c), and by the definitions,

$$\begin{aligned}1 \leqslant \frac{F_0(x)}{F_1(x)} &\leqslant \lim_{x\to\underline{\theta}} \frac{F_0(x)}{F_1(x)} \\ &= \frac{f_0(\underline{\theta})}{f_1(\underline{\theta})} && \text{(by l'Hôpital rule),} \\ &= \frac{1}{\underline{\theta}} && \text{(by the definition of } \theta\text{).}\end{aligned}$$

By (b), as $\frac{1-Q_t^0}{1-Q_t^1}$ is a convex combination of $\left\{\frac{F_0(\hat{x}_{t'})}{F_1(\hat{x}_{t'})}\right\}_{t'<t} \cup \{1\}$, and by (a),

$$\frac{k}{1-k} \leqslant \hat{x}_t \leqslant \frac{1}{\underline{\theta}} \frac{k}{1-k}$$

(e) By (d), Assumption 1(b), and the assumption that F_V has full support on $(\underline{\theta}, \overline{\theta})$, $0 = F_V(\underline{\theta}) < F_V(\frac{k}{1-k}) \leqslant F_V(\hat{x}_t) \leqslant F_V(\frac{1}{\underline{\theta}}\frac{k}{1-k}) < F_V(\overline{\theta}) = 1$. As $0 < \inf F_V(\hat{x}_t) < \sup F_V(\hat{x}_t) < 1$, $Q_t^V \in (0,1)$. □

A.1. Proofs in section 3

Proof of Lemma 1. Let γ_t be the probability that an agent has not found a problem until period t. Then, this satisfies $\gamma_{t+dt} = \gamma_t(1-\lambda_t dt)$. By solving this ODE, we obtain $\gamma_t = e^{-\Lambda(t)}$. The probability that the given agent finds the problem at the exact period t is $e^{-\Lambda(t)}\lambda_t dt$. As the agent announces the problem with probability $(1 - F_V(x_t))$,

$$Q_{t+dt}^V - Q_t^V = \underbrace{e^{-\Lambda(t)}\lambda_t}_{\text{prob. finds at } t} (1 - F_V(x_t))dt,$$

which completes the proof. □

Proof of Proposition 1. The payoff of announcement at period $t' > t \geqslant \tau_i$ is

$$\underbrace{[P_{t,\tau_i}(\theta_i)(1-(Q^1_{t'}-Q^1_t)) + (1-P_{t,\tau_i}(\theta))(1-(Q^0_{t'}-Q^0_t))]}_{\text{prob. The counterpart does not announce the problem at } \tau\in[t,t']} e^{-\rho(t'-t)}(P_{t',\tau_i}(\theta)-k).$$

By differentiation, $\frac{\partial P_{t',\tau_i}}{\partial t'} < 0$ if and only if

$$(1-F_1(\hat{x}_{t'}))[\alpha F_0(\hat{x}_{\zeta(\tau_i)}) + (1-\alpha)(1-Q^0_{t'})] > (1-F_0(\hat{x}_{t'}))[\alpha F_1(\hat{x}_{\zeta(\tau_i)})) + (1-\alpha)(1-Q^1_{t'}].$$

By monotonic likelihood ratio dominance property, F_1 first order stochastically dominates F_0. Therefore, $F_0(x) > F_1(x)$ for each x and $Q^1_t > Q^0_t$ for each t. Then, $P_{t',\tau_i}(\cdot) < P_{t,\tau_i}(\cdot)$, if $P_{t,\tau_i}(\theta) < k$, announcing after the period t is never optimal. Otherwise, announcing immediately is optimal.

Note that $P_{t,\tau_i}(\theta) > k$ if and only if

$$\theta > x_{t,\tau_i} := \frac{\alpha_\tau F_0(\hat{x}_{\zeta(\tau_i)}) + (1-\alpha_{\tau_i})(1-Q^0_t)}{\alpha_{\tau_i} F_1(\hat{x}_{\zeta(\tau_i)}) + (1-\alpha_{\tau_i})(1-Q^1_t)} \frac{k}{1-k}.$$

Then, the cutoff strategy with (x_{t,τ_i}) is an equilibrium. Moreover, as $P_{t,\tau_i}(\theta)$ decreases in t and increases in θ, we can demonstrate that x_{t,τ_i} is nondecreasing in t. Therefore, the cutoff strategy has IAP. By (1), we confirm that $\hat{x}_t = x_{t,t}$, which concludes the proof. □

Proof of Proposition 2. Note that as $\zeta(t) < t$, $\lim_{t\to\underline{T}} \zeta(t) = \underline{T}$. By abusing notation, define $\lim_{t\to\underline{T}} x_t = x_{\underline{T}}$.

Following Lemma 3 (a), recall that $\hat{x}_t$ is rewritten as follows:

$$\hat{x}_t = \left(\kappa_t \frac{F_0(\hat{x}_{\zeta(t)})}{F_1(\hat{x}_{\zeta(t)})} + (1-\kappa_t)\frac{1-Q^0_t}{1-Q^1_t}\right)\frac{k}{1-k}.$$

We focus on the case where $\hat{x}_t$ is differentiable; however, this does not lose generality.[12] By

[12]Even when $\hat{x}_t$ is not differentiable, it is sufficient to consider $\hat{x}_t - \hat{x}_{t-dt}$, in which case, by using the Taylor expan-

differentiating $\hat{x}_t$ with respect to t,

$$\begin{aligned}\dot{\hat{x}}_t = \Bigg[& \dot{\kappa}_t \left(\frac{F_0(\hat{x}_{\zeta(t)})}{F_1(\hat{x}_{\zeta(t)})} - \frac{1-Q_t^0}{1-Q_t^1} \right) \\ & + \kappa_t \frac{f_0(\hat{x}_{\zeta(t)})F_1(x_{\zeta(t)}) - F_0(\hat{x}_{\zeta(t)}) f_1(x_{\zeta(t)})}{(F_1(\hat{x}_{\zeta(t)}))^2} \dot{\hat{x}}_{\zeta(t)} \dot{\zeta}(t) \\ & + (1-\kappa_t) e^{-\Lambda(t)} \lambda_t \frac{-(1-F_0(\hat{x}_t))(1-Q_t^1) + (1-F_1(\hat{x}_t))(1-Q_t^0)}{(1-Q_t^1)^2} \Bigg] \frac{k}{1-k},\end{aligned} \tag{3}$$

(i) If $\zeta(t) = \underline{T}$ for each t, $\dot{\zeta}(t) = 0$, the second term is zero. Moreover, as Q_t^1 increases in t and $\dot{\alpha}_t \geqslant 0$, $\dot{\kappa}_t > 0$. By MHLP, the third term is positive. Then, consider the first term, as $\dot{\zeta} = 0$, $\dot{k}_t > 0$. Now we check whether

$$R(t) = \frac{F_0(\hat{x}_{\underline{T}})}{F_1(\hat{x}_{\underline{T}})} - \frac{1-Q_t^0}{1-Q_t^1} > 0$$

Note that $\lim_{t \to \underline{T}} R(t) > 0$ as $F_0(x_{\underline{T}}) > F_1(x_{\underline{T}})$. Suppose by contradiction that $R(t) \leqslant 0$ for some t. Let t^* be the smallest one of t. Then, as $\dot{x}(\tau) > 0$ for each $\tau < t^*$. Following Lemma 3 (b), $\frac{1-Q_t^0}{1-Q_t^1}$ is a convex combination of $\{\frac{F_0(\hat{x}_{t'})}{F_1(\hat{x}_{t'})}\}_{t'<t} \cup \{1\}$. As $\frac{F_0(x)}{F_1(x)}$ decreases in x (Lemma 3(c)), and $\hat{x}_\tau \geqslant \hat{x}_{\underline{T}}$ for each $\tau < t^*$, $\frac{1-Q_t^0}{1-Q_t^1} < \frac{F_0(\hat{x}_{\underline{T}})}{F_1(\hat{x}_{\underline{T}})}$. This implies that $R(t^*) > 0$, which is a contradiction.

(ii) If $\alpha_t = 0$ for each t, $\kappa_t = 0$ for each t. Therefore, the first and second terms of (3) are zero. Then, we only focus on the third term, which is positive.

(iii) Consider the case where $\alpha_t = 1$ for each t. In this case, the definition of $\hat{x}_t$ is

$$\hat{x}_t = \frac{F_0(\hat{x}_{\zeta(t)})}{F_1(\hat{x}_{\zeta(t)})} \frac{k}{1-k}.$$

sion, $\hat{x}_t - \hat{x}_{t-dt} = \dot{\kappa}_{t_1} dt \left(\frac{F_0(\hat{x}_{\zeta(t)})}{F_1(\hat{x}_{\zeta(t)})} - \frac{1-Q_t^0}{1-Q_t^1} \right) + \frac{\partial (F_0(\hat{x}_{\zeta(t_2)}))/(F_1(x_{\zeta(t_2)}))}{\partial t} (\hat{x}_{\zeta(t)} - \hat{x}_{\zeta(t-dt)}) + (1-\kappa_t) \frac{\partial (1-Q_{t_3}^0)/(1-Q_{t_3}^1)}{\partial t} dt$, where $t_1, t_2, t_3 \in [t-dt, t]$. Then, the similar logic to the following proof can be applied for the case.

Then, as $\lim_{s\to\infty} \zeta^s(t) = \underline{T}$, the above equation is rewritten as follows:

$$\hat{x}_t = \frac{F_0(\hat{x}_{\underline{T}})}{F_1(\hat{x}_{\underline{T}})} \frac{k}{1-k}.$$

Thus, $\hat{x}_t = \hat{x}_{\underline{T}}$ for each t. Then, $\hat{x}$ is constant. □

Proof of Proposition 3. We classify two cases: (a) Assumptions 1 and 2 are held and (b) Assumptions 1 and 3 are held.

First, we consider case (a). As in the proof of Proposition 2, we focus on the case where $\hat{x}_t$ is differentiable; however, this focus does not lose the generality.[13]

(i) We first show that for sufficiently small t, $\dot{\hat{x}}_t > 0$. Following Assumption 1 (a), for each $t < t^*$, $\dot{\zeta}(t) = 0$. Then, as in Proposition 2 (a), we show that $\dot{\hat{x}}_t \geqslant 0$ for each $t < t^*$.

(ii) Now, we show the existence of a decreasing point. Suppose by contradiction that $\dot{\hat{x}}_t \geqslant 0$ for each $t \in (\underline{T}, \overline{T})$.

Then, consider $\dot{\hat{x}}_t$, which is calculated as

$$\begin{aligned}\dot{\hat{x}}_t = \Bigg[&\dot{\kappa}_t \left(\frac{F_0(\hat{x}_{\zeta(t)})}{F_1(\hat{x}_{\zeta(t)})} - \frac{1-Q_t^0}{1-Q_t^1} \right) \\ &+ \kappa_t \frac{f_0(\hat{x}_{\zeta(t)})F_1(\hat{x}_{\zeta(t)}) - F_0(\hat{x}_{\zeta(t)})f_1(\hat{x}_{\zeta(t)})}{(F_1(\hat{x}_{\zeta(t)}))^2} \dot{x}_{\zeta(t)} \dot{\zeta}(t) \\ &+ (1-\kappa_t) e^{-\Lambda(t)} \lambda_t \frac{-(1-F_0(\hat{x}_t))(1-Q_t^1) + (1-F_1(\hat{x}_t))(1-Q_t^0)}{(1-Q_t^1)^2} \Bigg] \frac{k}{1-k},\end{aligned} \tag{4}$$

where

$$\begin{aligned}\dot{\kappa}_t = A\Big[&\dot{\alpha}_t F_1(\hat{x}_{\zeta(t)})(1-Q_t^1) \\ &+ \alpha_t(1-\alpha_t) f_1(\hat{x}_{\zeta(t)}) \dot{\hat{x}}_{\zeta(t)} \dot{\zeta}(t)(1-Q_t^1) + \alpha_t(1-\alpha_t) F_1(\hat{x}_{\zeta(t)})(1-F_1(\hat{x}_{\zeta(t)})) e^{-\Lambda(t)} \lambda_t \Big],\end{aligned}$$

and $A = (\alpha_t F_1(\hat{x}_{\zeta(t)}) + (1-\alpha_t)(1-Q_t^1))^{-2}$.

[13] See also footnote 12.

Consider the first term of (4). Note

$$\frac{F_0(\hat{x}_{\zeta(t)})}{F_1(\hat{x}_{\zeta(t)})} - \frac{1-Q^0_t}{1-Q^1_t} = -\frac{e^{-\Lambda(t)}}{1-Q^1_t} - \int_{\overline{T}}^{t} \frac{e^{-\Lambda(t')}\lambda_{t'}F_1(\hat{x}_{t'})}{1-Q^1_t}\frac{F_0(\hat{x}_{t'})}{F_1(\hat{x}_{t'})}dt' + \frac{F_0(\hat{x}_{\zeta(t)})}{F_1(\hat{x}_{\zeta(t)})}$$
$$= \begin{cases} \frac{e^{-\Lambda(t)}}{1-Q^1_t}(\varphi(\hat{x}_{\zeta(t)})-1) + \int_{\zeta(t)}^{t} \frac{e^{-\Lambda(t')}\lambda_t F_1(\hat{x}_{t'})}{1-Q^1_t}(\varphi(\hat{x}_{\zeta(t)})-\varphi(\hat{x}_{t'}))dt' \\ +\int_{\underline{T}}^{\zeta(t)} \frac{e^{-\Lambda(t')}\lambda_{t'}F_1(\hat{x}_{t'})}{1-Q^1_t}(\varphi(\hat{x}_{\zeta(t)})-\varphi(\hat{x}_{t'}))dt' \end{cases}, \tag{5}$$

where $\varphi(x) = \frac{F_0(x)}{F_1(x)}$. Note that $\int_{\zeta(t)}^{t} e^{-\Lambda(t')}\lambda_t F_V(\hat{x}_{t'})dt' < te^{-\Lambda(\zeta(t))}\sup_t \lambda_t$, which converges to 0 as $t \to \infty$. Therefore, the first and second terms of (5) converge to 0. Moreover, note that as x_t is nondecreasing and φ is decreasing, and that $\dot{\hat{x}}_t > 0$ for each $t \in [\underline{T}, t^*]$, the third term of (5) is negative. Note that by the definition of $\hat{x}$, $(\hat{x}_t)_{t<\overline{T}}$ does not depend on the size of $\overline{T}$. Therefore, if $\overline{T}$ is sufficiently large, there is $t \in (\underline{T}, \overline{T})$ such that $\frac{F_0(\hat{x}_{\zeta(t)})}{F_1(\hat{x}_{\zeta(t)})} - \frac{1-Q^0_t}{1-Q^1_t} < 0$. This value does not converges to 0 even when $\overline{T} \to \infty$; the third term of (5) is independent of $\overline{T}$. The convergent of $\frac{F_0(\hat{x}_{\zeta(t)})}{F_1(\hat{x}_{\zeta(t)})} - \frac{1-Q^0_t}{1-Q^1_t}$ is as follows:

$$\int_{\underline{T}}^{\infty} \frac{e^{-\Lambda(t')}\lambda_{t'}F_1(\hat{x}_{t'})}{1-Q^1_\infty}(\varphi(\hat{x}_\infty)-\varphi(\hat{x}_{t'}))dt'$$

Note that following Lemma 3 (c) and the supposition that $\hat{x}_t$ is nondecreasing, the second term of (4) is negative. Then, as we assume that $\dot{\hat{x}}_{t'} \geqslant 0$, for each $t' < t$, $\dot{\hat{x}}_t < 0$ if

$$A\Big[\dot{\alpha}_t F_1(\hat{x}_{\zeta(t)})(1-Q^1_t) + \alpha(1-\alpha)F_1(\hat{x}_{\zeta(t)})(1-F_1(\hat{x}_{\zeta(t)}))e^{-\Lambda(t)}\lambda_t\Big]\left(\frac{F_0(\hat{x}_{\zeta(t)})}{F_1(\hat{x}_{\zeta(t)})} - \frac{1-Q^0_t}{1-Q^1_t}\right)$$
$$< (1-\kappa_t)e^{-\Lambda(t)}\lambda_t \frac{(1-F_0(\hat{x}_t))(1-Q^1_t)-(1-F_1(\hat{x}_t))(1-Q^0_t)}{(1-Q^1_t)^2}.$$

As $\dot{\alpha}_t > 0$, by dividing $\dot{\alpha}_t$ and letting $t \to \infty$, the above inequality becomes

$$A\Big[F_1(\hat{x}_\infty)(1-Q^1_\infty)\Big]\left(\frac{F_0(\hat{x}_\infty)}{F_1(\hat{x}_\infty)} - \frac{1-Q^0_\infty}{1-Q^1_\infty}\right) < 0. \tag{6}$$

Note that as x_t is monotone and bounded, $\lim_{t\to\infty} \hat{x}_{\zeta(t)} = \lim_{t\to\infty} \hat{x}_t$.

Inequality (6) is satisfied because $\frac{F_0(\hat{x}_\infty)}{F_1(\hat{x}_\infty)} - \frac{1-Q^0_\infty}{1-Q^1_\infty} < 0$, as shown above. Therefore, for sufficiently large t, $\dot{\hat{x}}_t < 0$, which is a contradiction.

Next, we consider case (b). Suppose by contradiction that $\dot{\hat{x}}_t \geqslant 0$. As in the proof of (a), because $\dot{\zeta}_{t'} = 0$ for each t', $\dot{\hat{x}}_t < 0$ if

$$A\Big[\dot{\alpha}_t F_1(\hat{x}_{\zeta(t)})(1-Q^1_t) + \alpha(1-\alpha)F_1(\hat{x}_{\zeta(t)})(1-F_1(\hat{x}_{\zeta(t)}))e^{-\Lambda(t)}\lambda_t\Big]\left(\frac{F_0(\hat{x}_{\zeta(t)})}{F_1(\hat{x}_{\zeta(t)})} - \frac{1-Q^0_t}{1-Q^1_t}\right)$$
$$< (1-\kappa_t)e^{-\Lambda(t)}\lambda_t \frac{(1-F_0(\hat{x}_t))(1-Q^1_t) - (1-F_1(\hat{x}_t))(1-Q^0_t)}{(1-Q^1_t)^2}.$$

As $\dot{\alpha}_t < 0$, by dividing $\dot{\alpha}_t$ and letting $t \to \infty$, the above inequality becomes

$$A\Big[F_1(\hat{x}_\infty)(1-Q^1_\infty)\Big]\left(\frac{F_0(\hat{x}_\infty)}{F_1(\hat{x}_\infty)} - \frac{1-Q^0_\infty}{1-Q^1_\infty}\right) > 0. \tag{7}$$

Note that by $\zeta(t) = \underline{T}$, as in the proof of Proposition 2, $\frac{F_0(\hat{x}_\infty)}{F_1(x_\infty)} - \frac{1-Q^0_\infty}{1-Q^1_\infty} > 0$. Therefore, (7) is satisfied. This implies that $\dot{\hat{x}}_t < 0$ and thus it is a contradiction. □

A.2. Proofs in Section 4

Proof of Lemma 2. Let γ_t be the probability that an agent has not found a problem until period t. Then, this satisfies $\gamma_{t+dt} = \gamma_t(1-\lambda_t dt)$. By solving this ODE, we obtain $\gamma_t = e^{-\Lambda(t)}$. As in Lemma 1, the probability that the given agent finds the problem at the exact period t' is $e^{-\Lambda(t')}\lambda_{t'}dt'$. By period t, the agent announces the problem if and only if $\theta > \tilde{x}_{t''}$ for some $t'' \in [t', t]$. As the probability that the agent announces the problem is $(1 - F_V(\xi_t(t')))$,

$$\tilde{Q}^V_{t+dt} - \tilde{Q}^V_t = \underbrace{e^{-\Lambda(t')}\lambda_{t'}}_{\text{prob. finds at } t'}(1-F_V(\xi_t(t')))dt',$$

which completes the proof. □

Proof of Proposition 4. Note that under the belief of agent i at period $t \geqslant \tau_i$, the probability of $V = 1$ under the condition that no announcement occurs is

$$\tilde{P}_{t',t}(\theta_i) := \frac{\theta_i(\alpha_t F_1(\xi_t(\zeta(t)) + (1-\alpha_t)(1-\tilde{Q}^1_{t'}))}{\theta_i(\alpha_t F_1(\xi_t(\zeta(t)) + (1-\alpha_t)(1-\tilde{Q}^1_{t'})) + (\alpha_t F_0(\xi_t(\zeta(t)) + (1-\alpha_t)(1-\tilde{Q}^0_{t'}))}.$$

This comes from the agents' anticipation that the bias never changes in the future.

Then, as in the proof of Proposition 1, under the belief of agent i at period t, the payoff of announcing at $t' > t \geqslant \tau_i$ is

$$\underbrace{[\tilde{P}_{t,t}(\theta_i)(1-(\tilde{Q}^1_{t'}-\tilde{Q}^1_t)) + (1-\tilde{P}_{t,t}(\theta))(1-(\tilde{Q}^0_{t'}-\tilde{Q}^0_t))]}_{\text{prob. The other announces the problem at some } t''\in[t,t']} e^{-\rho(t'-t)}(\tilde{P}_{t',t}(\theta)-k).$$

By differentiation, $\frac{\partial \tilde{P}_{t',t}}{\partial t'} < 0$ if and only if

$$(1-F_1(\tilde{x}_{t'}))[\alpha F_0(\xi(\zeta(t))) + (1-\alpha)(1-\tilde{Q}^0_{t'})] > (1-F_0(\tilde{x}_{t'}))[\alpha F_1(\xi(\zeta(t))) + (1-\alpha)(1-\tilde{Q}^1_{t'})].$$

This inequality holds as in the proof of Proposition 1. Therefore, if $\tilde{P}_{t,t} = \tilde{P}_t > k$, announcing immediately is optimal. Otherwise, as agent i anticipates that $\tilde{P}_{t',t} < k$ and they believe that they will never announce in the future, not announcing is optimal. As $\tilde{P}_t \geqslant k$ if and only if $\theta_i \geqslant \tilde{x}_t$, we conclude the proof. □

Proof of Proposition 5. (a) Suppose that the sequence $(\hat{x}_t)$ satisfies (1) and is nondecreasing. As $\hat{x}_t$ is nondecreasing, $\min_{t'\in[s,t]} \hat{x}_{t'} = \hat{x}_s$. Then, $\tilde{P}_t(\theta) = P_{t,t}(\theta)$ with the cutoff strategy $(\hat{x}_t)$ under moving projection. Therefore, $\hat{x}_t$ satisfies (2). Following Proposition 4, there is equilibrium with $(\tilde{x}_t) = (\hat{x}_t)$. If $\theta > \tilde{x}_t$, as shown in Proposition 1, announcing immediately is optimal. Otherwise, as $\tilde{x}_{t'} > \tilde{x}_t \geqslant \theta$, announcing is never optimal in later period. Therefore, the cutoff strategy is at an equilibrium state and has IAP.

(b) Suppose by contradiction that the cutoff strategy $(\tilde{x}_t)$ having IAP is at an equilibrium state. In this case, $\tilde{P}_t(\theta) = P_{t,t}(\theta)$. Then, as the sequence satisfying (1) is nonmonotonic,

$\tilde{x}_t$ is also nonmonotonic. Recall that the (current) benefit of announcement is $P_{t,t}(\theta) > k$, equivalent to $\theta > \tilde{x}_t$. As $\tilde{x}_t$ is nonmonotonic, there is t, t' with $t > t'$ and θ such that $\tilde{x}_{t'} > \theta > \tilde{x}_t$. Then, under moving projection, at time t', the payoff of announcing is negative. As shown in the proof of Proposition 4, the agent believes that they will never announce in the future. Then, not announcing is optimal; however, at period t, the payoff of the announcement is positive. Therefore, $a_{it} = 1$ becomes optimal, proving that the cutoff strategy $(\tilde{x}_t)_t$ cannot have IAP. □

Proof of Theorem 1. As $\tilde{x}_t$ is nonmonotonic, there are t, t' with $t' > t$ such that $\tilde{x}_t > \tilde{x}_{t'}$. When an agent finds a problem at t and receives $\theta \in (\tilde{x}_t, \tilde{x}_{t'})$, then, under the cutoff strategy with $\tilde{x}_t$, they decide not to announce at t, but decide to announce at $t'' = \inf\{s \geqslant t : \tilde{x}_s \leqslant \theta\} < t'$. □

B. Formal analyses in Section 5

In this section, we formally analyze a few topics listed in Section 5. First, we perform comparative statics concerning α. The following proposition reveals that with any α, the cutoff at period $\underline{T}$ is lower than under full projection (i.e., $\alpha = 1$), and at a sufficiently later period, the cutoff is higher that under full projection.

Proposition 6. *Suppose that* $\lim_{t\to\infty} \Lambda(t) = \infty$. *Consider* α *with* $\lim_{t\to\infty} \alpha_t < 1$. *Let,* $\hat{x}^{\alpha}_t$ *be the cutoff with such* α. *Moreover, denote* $\hat{x}^1_t$ *be the cutoff when* $\alpha_t = 1$ *for each* t. *Then, if* $\hat{x}_t$ *increases,* $\lim_{t\to\infty} \hat{x}^{\alpha}_t > \hat{x}^1_t > \hat{x}^{\alpha}_{\underline{T}}$.

Proof of Proposition 6. Following Proposition 2 (b), if $\alpha_t = 1$ for each t, $\hat{x}^1_t$ is constant, and therefore, we drop t. $\hat{x}^1$ satisfies

$$\hat{x}^1 = \frac{F_0(\hat{x}^1)}{F_1(\hat{x}^1)} \frac{k}{1-k}.$$

As Lemma 3 (c) shows, $F_0(x)/F_1(x)$, $\hat{x}^1$ is uniquely determined.

Next, consider $\hat{x}_t^{\alpha}$. If $t = \underline{T}$,

$$\hat{x}_{\underline{T}}^{\alpha} = \frac{\alpha_{\underline{T}} F_0(\hat{x}_{\underline{T}}^{\alpha}) + (1-\alpha_{\underline{T}})}{\alpha_{\underline{T}} F_1(\hat{x}_{\underline{T}}^{\alpha}) + (1-\alpha_{\underline{T}})} \frac{k}{1-k}.$$

Note that for each x,

$$\frac{F_0(x)}{F_1(x)} \geqslant \frac{\alpha_{\underline{T}} F_0(x) + (1-\alpha_{\underline{T}})}{\alpha_{\underline{T}} F_1(x) + (1-\alpha_{\underline{T}})}.$$

Therefore, $\hat{x}^1 > \hat{x}_{\underline{T}}^{\alpha}$.

Finally, consider the case $t \to \infty$. Suppose by contradiction that $\hat{x}^1 > \lim_{t\to\infty} \hat{x}_t^{\alpha}$. As $\hat{x}_t^{\alpha}$ increases in t, $\hat{x}^1 > \hat{x}_t^{\alpha}$ for each t. Moreover, as $\frac{1-Q_t^0}{1-Q_t^1}$ is written as

$$\frac{1-Q_t^0}{1-Q_t^1} = \frac{1-\int_{\underline{T}}^t e^{-\Lambda(t')}\lambda_{t'} dt'}{1-Q_t^1} + \int_{\underline{T}}^t \frac{e^{-\Lambda(t')}\lambda_{t'} dt' F_1(\hat{x}_{t'}^{\alpha})}{1-Q_t^1} \frac{F_0(\hat{x}_{t'}^{\alpha})}{F_1(\hat{x}_{t'}^{\alpha})},$$

and $\Lambda(t) \to \infty$,

$$\lim_{t\to\infty} \frac{1-Q_t^0}{1-Q_t^1} = \int_{\underline{T}}^{\infty} \frac{e^{-\Lambda(t')}\lambda_{t'} dt' F_1(\hat{x}_{t'}^{\alpha})}{1-Q_{\infty}^1} \frac{F_0(\hat{x}_{t'}^{\alpha})}{F_1(\hat{x}_{t'}^{\alpha})}.$$

then, $\lim_{t\to\infty} \hat{x}_t^{\alpha}$ converges to a convex combination of $\{\frac{F_0(\hat{x}_{t'}^{\alpha})}{F_1(\hat{x}_{t'}^{\alpha})}\}$. As $F_0(x)/F_1(x)$ decreases in x and $\hat{x}^1 > \hat{x}_t^{\alpha}$ for each t, $\lim_{t\to\infty} \hat{x}_t^{\alpha} > \hat{x}^1$, which is a contradiction. □

Next, we consider the issue that agents are rational about future biases changing under moving projection. In this case, the following proposition shows that the cutoff strategy may not be at an equilibrium state.

Proposition 7. *Suppose that each agent correctly anticipates their bias. When ρ is small enough, there does not exist equilibrium with a cutoff strategy satisfying the following properties: i) $\theta_{t,t,\tau} = \tilde{x}_t$, ii) $\tilde{x}_t$ has a decreasing region, and iii) $\tilde{x}_t$ is continuously differentiable.*

Proof of Proposition 7. Suppose by contradiction that such an equilibrium exists. Suppose also that $\dot{\tilde{x}}_t > 0$ for each $t \in\in [t_1, t_2)$ and $\dot{\tilde{x}}_t < 0$ for each $t \in (t_2, t_3]$. Consider θ^* that satisfy $\theta^* = \tilde{x}_{t_2}$. Then, $\tilde{P}_{t_2}(\theta^*) = k$.

For each agent with $\theta < \theta^*$, the payoff of announcing at time t_2 is negative. Consider such an agent who decides whether to announce at time $t = t_2$. The payoff of announcing at time t', which is

$$e^{-\rho(t'-t_2)} \Pr|_{\beta_{t_2}^{t_2}}(a_{tj} = 0, \forall t < t' \mid a_{tj} = 0, \forall t < t_2)(\tilde{P}_{t',t_2}(\theta) - k),$$

where

$$\begin{aligned}
&\Pr|_{\beta_{t_2}^{t_2}}(a_{tj} = 0, \forall t < t' \mid a_{tj} = 0, \forall t < t_2) \\
&= \frac{\theta(\alpha_{t_2} F_1(\xi_{t_2}(\zeta(t_2)) + (1-\alpha_{t_2})(1-\tilde{Q}^1_{t'})) + (\alpha_{t_2} F_0(\xi_t(\zeta(t)) + (1-\alpha_{t_2})(1-\tilde{Q}^0_{t'}))}{\theta(\alpha_{t_2} F_1(\xi_{t_2}(\zeta(t_2)) + (1-\alpha_{t_2})(1-\tilde{Q}^1_{t_2})) + (\alpha_{t_2} F_0(\xi_{t_2}(\zeta(t_2)) + (1-\alpha_{t_2})(1-\tilde{Q}^0_{t_2}))}, \\
&\tilde{P}_{t',t_2}(\theta) := \frac{\theta(\alpha_{t_2} F_1(\xi_{t_2}(\zeta(t_2))) + (1-\alpha_{t_2})(1-\tilde{Q}^1_{t'}))}{\theta(\alpha_{t_2} F_1(\xi_{t_2}(\zeta(t_2)) + (1-\alpha_{t_2})(1-\tilde{Q}^1_{t'})) + (\alpha_{t_2} F_0(\xi_{t_2}(\zeta(t_2)) + (1-\alpha_{t_2})(1-\tilde{Q}^0_{t'}))}.
\end{aligned}$$

Then, note that

$$(\tilde{P}_{t_2}(\theta) - k) > \Pr|_{\beta_{t_2}^{t_2}}(a_{tj} = 0, \forall t < t' \mid a_{tj} = 0, \forall t < t_2)(\tilde{P}_{t',t_2}(\theta) - k),$$

if and only if

$$\begin{aligned}
&\theta(\alpha_{t_2} F_1(\xi_{t_2}(\zeta(t_2))) + (1-\alpha_{t_2})(1-\tilde{Q}^1_{t_2})) - k\left(\theta(\alpha_{t_2} F_1(\xi_{t_2}(\zeta(t_2)) + (1-\alpha_{t_2})(1-\tilde{Q}^1_{t_2})) + (\alpha_{t_2} F_0(\xi_{t_2}(\zeta(t_2)) + (1-\alpha_{t_2})(1-\tilde{Q}^0_{t_2}))\right) \\
&> \theta(\alpha_{t_2} F_1(\xi_{t_2}(\zeta(t_2))) + (1-\alpha_{t_2})(1-\tilde{Q}^1_{t'})) - k\left(\theta(\alpha_{t_2} F_1(\xi_{t_2}(\zeta(t_2)) + (1-\alpha_{t_2})(1-\tilde{Q}^1_{t'})) + (\alpha_{t_2} F_0(\xi_{t_2}(\zeta(t_2)) + (1-\alpha_{t_2})(1-\tilde{Q}^0_{t'}))\right).
\end{aligned}$$

If t' is sufficiently close to t_2, this inequality is satisfied if

$$\theta > \frac{\left[(1 - F_0(\tilde{x}_{t'})) - \int_{t_2}^{t'} e^{\Lambda(t)} \lambda_t f_0(\tilde{x}_{t'}) \dot{\tilde{x}}_{t'} dt\right]}{\left[(1 - F_1(\tilde{x}_{t'})) - \int_{t_2}^{t'} e^{\Lambda(t)} \lambda_t f_1(\tilde{x}_{t'}) \dot{\tilde{x}}_{t'} dt\right]} \frac{k}{1-k} \tag{8}$$

As $1-F_1 \geqslant 1-F_0$, the right-hand side is smaller than $\tilde{x}_t$ for each t when t' is sufficiently small. Consider $\theta < \theta^* = \tilde{x}_{t_2}$ that satisfies $\tilde{x}_{t'} = \theta$. Then, the agent receiving θ announces at time t'. If $\rho = 0$ and θ is sufficiently close to θ^*, as (8) is satisfied, the expected utility of announcing at time t_2 is greater than that of announcing at time t'. Then, when $\tilde{P}_{t_2}(\theta) < k$, the agent is incentivized to deviate by announcing at period t_2. □

C. The existence of the equilibrium

This section proves the existence of $\hat{x}$ and $\tilde{x}$, defined in (1) and (2).

Proposition 8. *Assume Assumption 1. Suppose that ζ is continuous. Then, $(\hat{x}_t)$ that satisfying (1) exists. Moreover, $\hat{x}_t$ is continuous in t.*

Proof of Proposition 8. Define functional G as

$$G[\hat{x}](t) = \frac{\alpha_t F_0(\hat{x}_{\zeta(t)}) + (1-\alpha_t)(1-Q_t^0)}{\alpha_t F_1(\hat{x}_{\zeta(t)}) + (1-\alpha_t)(1-Q_t^1)} \frac{k}{1-k}.$$

Then, $\hat{x}$ is a fixed point of G. The remainder of this proof demonstrates the existence of the fixed point.

Following Lemma 3(d) $G[\hat{x}](t) \in (\underline{\theta}, \overline{\theta})$. Furthermore, if $\hat{x}_t$ is continuous in t, $G[\hat{x}](t)$ is continuous in t.

Let $C^1 = \{h \in [\underline{\theta}, \overline{\theta}]^{[\underline{T}, t^*]} : h \text{ is continuous}\}$. Then, C^1 is convex and nonempty. Now we show that the restriction of G to $[\underline{T}, t^*]$, $G|_{[\underline{T},t^*]}[C^1]$ is well-defined self map on C^1, bounded, and equicontinuous. As Q_t^V depends on $(\hat{x}_\tau)_{\tau<t}$ but is independent of $(\hat{x}_\tau)_{\tau>t}$, $G|_{[\underline{T},t^*]}[C^1], (t)$ is well-defined for each $t \leqslant t^*$. Then, $G|_{[\underline{T},t^*]}[C^1] \subseteq C^1$. Boundedness is already shown above. We prove the equicontinuity. As $\zeta(t) = \underline{T}$ for each $t \leqslant t^*$, by using Taylor expansion,

$$G[\hat{x}](t) - G[\hat{x}](t') = \Bigg[\dot{\kappa}_{t_1}(t-t')\left(\frac{F_0(\hat{x}_{\underline{T}})}{F_1(\hat{x}_{\underline{T}})} - \frac{1-Q_t^0}{1-Q_t^1}\right) \\ + (1-\kappa_t)\, e^{-\Lambda(t)}\lambda_t \frac{-(1-F_0(\hat{x}_t))(1-Q_t^1) + (1-F_1(\hat{x}_t))(1-Q_t^0)}{(1-Q_t^1)^2}(t-t')\Bigg]\frac{k}{1-k},$$

where $t_1, t_3 \in (t, t')$,

$$\dot{\kappa}_t = A\Big[\dot{\alpha}_t F_1(\hat{x}_{\underline{T}})(1-Q_t^1) + \alpha_t(1-\alpha_t)F_1(\hat{x}_{\underline{T}})(1-F_1(\hat{x}_{\underline{T}}))e^{-\Lambda(t)}\lambda_t\Big],$$

and $A = (\alpha_t F_1(\hat{x}_{\zeta(t)}) + (1-\alpha_t)(1-Q_t^1))^{-2}$. Note that by MLHP, $F_0 \geqslant F_1$, and $Q_t^1 \geqslant Q_t^0$. Following Lemma 3, $Q_t^1 < 1$ for each t and V. Then, there is a positive number, M, which is independent of $\hat{x}$, such that $|G[\hat{x}](t) - G[\hat{x}](t')| \leqslant M|t-t'|$. Then, $G|_{[\underline{T},t^*]}[C^1]$ is equicontinuous. By the Arzelá and Ascoli theorem, $G|_{[\underline{T},t^*]}[C^1]$ is compact. Then, the Schauder's fixed point theorem[14] implies that $G|_{[\underline{T},t^*]}$ has a fixed point. Let $\hat{x}^1 = (\hat{x}_t^1)_{[\underline{T},t^*]}$ be the fixed point.

The remainder of the proof constructs the fixed point of the restriction of G to partitioned subintervals of $[\underline{T}, \overline{T}]$.

For each $n = 2, 3, \ldots$, let $\psi(t) := \zeta^{-1}(t)$, $\psi^n(t) = \psi(\psi^{n-1}(t))$ for each $n \in \mathbb{N}$, ψ^0 be identity, and define

$$C^2 = \left\{h \in [\underline{\theta}, \overline{\theta}]^{[\psi^0(t^*),\psi^1(t^*)]} : h \text{ is continuous with } h_{\psi^0(t^*)} = x^1_{\psi^0(t^*)}\right\}.$$

We construct the fixed point of $G|_{[\psi^0(t^*),\psi^1(t^*)]}$ with fixing $(\hat{x}_t)_{t\leqslant t^*} = (\hat{x}_t^1)_{t\leqslant t^*}$. Using a similar method, we demonstrate that $G|_{[\psi^0(t^*),\psi^1(t^*)]}$ is a well-defined self map on C^2 and bounded. The equicontinuity is shown as follows. Consider $h \in C^2$. Then, by the Taylor expansion we

[14] See, e.g., Ok (2006), p. 627.

have that

$$G|_{[\psi^0(t^*),\psi^1(t^*)]}[h](t) - G|_{[\psi^0(t^*),\psi^1(t^*)]}[h](t') = \Bigg[\dot{\kappa}_{t_1}(t-t')\left(\frac{F_0(\hat{x}^1_{\zeta(t)})}{F_1(\hat{x}^1_{\zeta(t)})} - \frac{1-Q^0_t}{1-Q^1_t}\right)$$
$$+ \frac{\partial(F_0(\hat{x}^1_{\zeta(t_2)}))/(F_1(\hat{x}^1_{\zeta(t_2)}))}{\partial t}(\hat{x}^1_{\zeta(t)} - \hat{x}^1_{\zeta(t')}) + (1-\kappa_t)\frac{\partial(1-Q^0_{t_3})/(1-Q^1_{t_3})}{\partial t}(t-t')\Bigg]\frac{k}{1-k},$$

where $t_1, t_2, t_3 \in (t, t')$ and

$$\dot{\kappa}_t = A\Bigg[\dot{\alpha}_t F_1(\hat{x}_{\zeta(t)})(1-Q^1_t) + \alpha_t(1-\alpha_t)f_1(\hat{x}_{\zeta(t)})\frac{\hat{x}^1_{\zeta(t)} - \hat{x}^1_{\zeta(t')}}{t-t'}(1-Q^1_t)$$
$$+\alpha_t(1-\alpha_t)F_1(\hat{x}_{\zeta(t)})(1-F_1(\hat{x}_{\zeta(t)}))e^{-\Lambda(t)}\lambda_t\Bigg].$$

As $t < \psi^1(t^*) = \zeta^{-1}(t^*)$, $\zeta(t) < t^*$. Then, $\hat{x}^1_{\zeta(t)}$ is independent of the choice of $h \in C^2$. Then, as $\hat{x}^1$ and ζ are continuous, we also show a positive number, M, which is independent of h, such that $|G[h](t) - G[h](t')| \leqslant M|t-t'|$. This shows the equicontinuity of $G|_{[\psi^0(t^*),\psi^1(t^*)]}[C_2]$. Applying the same method for $n = 1$, we can show the existence of the fixed point $x^2 = (x_t)_{t\in[\psi^0(t^*),\psi^1(t^*)]}$.

Now for each n, define

$$C^n = \left\{h \in [\underline{\theta}, \overline{\theta}]^{[\psi^{n-2}(t^*),\psi^{n-1}(t^*)]} : h \text{ is continuous with } h_{\psi^{n-2}(t^*)} = \hat{x}^{n-1}_{\psi^{n-2}(t^*)}\right\}.$$

By applying the same method for $n = 2$, we can construct $\hat{x}^n = (\hat{x}_t)_{t\in[\psi^{n-2}(t^*),\psi^{n-1}(t^*)]} \in C^n$ that satisfies $\hat{x}^n = G|_{[\psi^{n-2}(t^*),\psi^{n-1}(t^*)]}[\hat{x}^n]$ for each $n \in \mathbb{N}$. Then, defining $\hat{x} = (\hat{x}^1, \hat{x}^2, \ldots)$, $\hat{x} = G[\hat{x}]$, which concludes the proof. □

Proposition 9. *Assume Assumption 1. Suppose that $\frac{1}{\underline{\theta}}\frac{k}{1-k} < \overline{\theta}$. Then, $(\tilde{x}_t)$ that satisfies (2) exists.*

Proof. Define

$$\tilde{G}[\tilde{x}](t) = \frac{\alpha_t F_0(\xi_t(\zeta(t))) + (1-\alpha_t)(1-\tilde{Q}_t^0)}{\alpha_t F_1(\xi_t(\zeta(t))) + (1-\alpha_t)(1-\tilde{Q}_t^1)} \frac{k}{1-k}.$$

Let $C = \left\{ h \in [\underline{\theta}, \overline{\theta}]^{[\underline{T}, \overline{T}]} \right\}$. As in the proof of Proposition 8, $\tilde{G}$ is a self map on C. Moreover, $\tilde{G}$ is continuous with product topology. As C is convex and compact with product topology (Tychonoff theorem), a fixed point of $\tilde{G}$ exists. □